

POLARIZED LIGHT IN THE STUDY OF ORES AND METALS.¹

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The measurement of the optical properties of transparent minerals, even in minute, irregular grains, is a simple task with modern petrographic microscope methods and is accomplished by petrologists as part of ordinary routine work. But the determination of the optical constants of opaque substances is difficult and is rarely attempted by microscopists; all observations are necessarily made in reflected light and are restricted commonly to the determination of color, of the character of crystallization, and of the behavior of the mineral or metal plate toward reagents and abrasives. It is generally recognized that if methods were available by means of which the optical constants of opaque substances in fine particles could be readily ascertained these methods would be of great value not only to students of ores and opaque minerals, but especially to metal-

¹ The manuscript of this paper was finished in March, 1917, and is here presented without alteration. A brief resumé of the results of the investigation was given at the meeting of the American Philosophical Society in April, 1917. With our entrance into the war the writer joined the Army, and the publication of this paper was accordingly postponed.

In the theoretical section of this paper certain standard equations are derived and expressed in Cartesian coördinates. The expressions would have been much simpler and shorter had the methods of vector analysis been employed; but this was not done and the equations are developed in the usual notation in order that they may be easily accessible to the reader interested in this particular subject. Many of the problems of crystal optics are, however, essentially vectorial in character and yield most readily, as do many problems in electricity (alternating currents, wireless telegraphy), to treatment by vector analysis. In vector analysis the imaginary quantity $i = \sqrt{-1}$ is treated simply as an operator rotating a vector through 90°. This greatly simplifies the interpretation of equations containing complex quantities. An interpretation of this kind of many of the equations in the present paper would undoubtedly render them more intelligible, but this would have greatly increased the length of the paper and was accordingly not attempted.

lographers in the study of metal alloys. Unfortunately the explanation of the phenomena presented by opaque bodies is somewhat involved and the experimental measurement of the optical constants of such substances is encumbered with difficulties which, in many cases, allow only approximate results to be obtained at best. In view of these facts it has seemed to the writer that a useful purpose may be served by a critical discussion, based on the commonly accepted version of the electromagnetic theory of light for absorbing bodies, not only of the phenomena which might be of value for diagnostic purposes, but also of the factors underlying the several methods, old and new, which may be applied to the measurement of the few determinable optical properties of opaque bodies. A proper appreciation of the possibilities and also of the limitations of the application of polarized light to the study of opaque substances can only be had by a proper understanding of the fundamental principles involved and of their relative significance in practical diagnosis.

In the theoretical treatment of the general problem emphasis will be placed on the phenomena resulting on perpendicular reflection (angle of incidence = 0), because most metallographic observations with the microscope are made under conditions of vertical illumination. The attempt will be made to present the fundamentals of the subject and to indicate the mode of derivation of the equations required. Although several new relations are given, the treatment as a whole is necessarily along lines which have been followed by others. In the preparation of the section on theory the following books and articles have been specially consulted: P. Drude in Winkelmann's "Handbuch der Physik," Vol. VI, and in *Annalen der Physik*; "Lehrbuch der Optik," by P. Drude; "Lehrbuch der Kristallphysik," by F. Pockels; "Physical Optics," by A. Schuster; "The Analytical Theory of Light," by J. Walker; and "Physical Optics," by R. W. Wood.

The results of the investigation may be summarized by stating that in general the optical constants, such as refractive indices and absorption indices, cannot be satisfactorily ascertained on small polished random sections; that the application of polarized light enables the observer ordinarily to determine whether the crystal plate is isotropic or anisotropic and also to ascertain the degree of aniso-

tropism; that for this determination methods based either on the contrast in intensity of the two reflected components or on the amount of rotation of the plane of polarization on reflection of the incident plane-polarized light may be employed; that methods based on the phase difference between the two reflected components are in general of little value because of the small differences in phase which ordinarily result for a relatively large change in birefringence or biabsorption.

Several new methods are described for detecting anisotropism in opaque substances; of these, that requiring the use of the bi-quartz-wedge-plate is the simplest and has been found in practice to be superior to any heretofore suggested.

THEORETICAL.

Light waves on passing through homogeneous material are absorbed to a greater or less degree. In transparent media the amount of absorption (in the visible spectrum) is relatively slight and can be neglected for most purposes; but in absorbing media there is an appreciable weakening in intensity even in thin plates, which gives rise to the phenomena of absorption. No body is, however, perfectly opaque (perfect absorber) or perfectly transparent, and the terms transparent and absorbing are relative terms which express, in a general way, the degree of absorption in the visible spectrum. The conditions may, of course, be reversed in the infra-red or ultra-violet. The experimental law of absorption, as expressed by Lambert, states that in a homogeneous medium each layer of equal thickness absorbs an equal fraction of the light transmitted; if the layers be considered to be one molecule deep, as in some crystals, then each layer absorbs the same percentage of the light which passes through it. In other words, for a layer of given thickness the intensity of light transmitted (total light less absorbed light) is proportional to the intensity of the incident light or

$$\frac{dI}{dl} = -I. \quad I = I_0 e^{-ml}, \quad (1)$$

where I_0 = intensity of incident light, I , intensity of transmitted light, l , thickness of layer, and m , the absorption modulus (*i.e.*, the

density of a layer of unit thickness²); since the intensity varies with the square of the amplitude of vibration of the transmitted light, the equation is valid

$$A = A_0 e^{-\frac{2\pi}{\lambda_0} \cdot k \cdot l} = A_0 e^{-\frac{2\pi}{\lambda} \cdot \kappa \cdot l}, \quad (2)$$

in which A_0 is the amplitude *in vacuo*, A , amplitude in the medium after passage through thickness l , λ , wave-length in material, λ_0 , wave-length *in vacuo*, k , the absorption coefficient, κ , the absorption index. But^{*}

$$\frac{A^2}{A_0^2} = \frac{I}{I_0} = e^{-\frac{4\pi}{\lambda} \cdot \kappa \cdot l} = e^{-\frac{4\pi}{\lambda_0} \cdot k \cdot l} = e^{-ml}; \quad (2a)$$

therefore

$$k = n\kappa \quad \text{and} \quad m = \frac{4\pi k}{\lambda_0} = \frac{4\pi \kappa}{\lambda}.$$

The significance of the absorption index can best be realized by a transformation of equation (2a)

$$\ln \frac{I_0}{I} = \frac{4\pi}{\lambda_0} l \cdot n \cdot \kappa$$

or expressed in ordinary logarithms

$$\log_{10} \frac{I_0}{I} = \frac{4\pi \cdot 0.43429}{\lambda_0} \cdot l \cdot n \cdot \kappa.$$

² The significance of these terms is clearly expressed in photography in which the relative densities of the photographic plate serve to record more or less accurately the relative intensities of the light impinging upon it. The incident light produces changes of such nature in the silver bromide particles in the emulsion that after development the exposed particles remain as opaque specks of silver. The more intense the incident light, the greater the number of affected silver particles. In the developed plate the more silver specks there are per given area, the more light is stopped, and the more opaque and dense is the plate. Density is measured in number of silver particles per unit area (total weight of silver particles per unit area). If a layer is of such density that it transmits the fraction a of the incident light, then two such layers superimposed transmit $a \cdot a$ or a^2 of the light, three layers $a \cdot a \cdot a$ or a^3 ; thus the density increases in arithmetical, but the transmission decreases in geometrical progression. If I/I_0 = percentage of light transmitted ($= e^{-lm}$ equation (1)) the reciprocal, I_0/I is the opacity $O = e^{ml}$ and the density $D = \log$ opacity or ml or simply m for $l = 1$. A plate is said to have unit density $D = 1 = \log$ opacity $= \log I_0/I = \log 10$, *i.e.*, when it transmits only $1/10$ of the incident light. Similarly a crystal plate of absorption-modulus 1 and of unit thickness transmits only $1/10$ of the incident monochromatic light.

For a wave length $\lambda_0 = .0005458$ mm. (near the mercury green line) this expression reduces to

$$\log_{10} \frac{I_0}{I} = 10000 \cdot l \cdot n \cdot \kappa. \quad (2b)$$

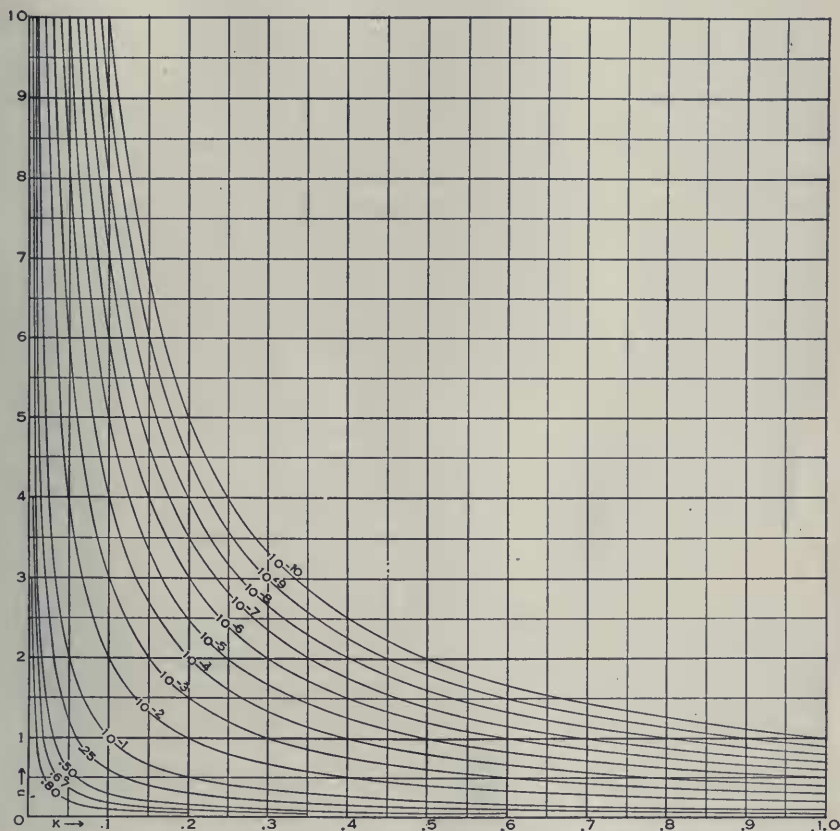


FIG. 1. In this figure the curves are equi-intensity curves indicating the ratio of the intensity of the transmitted light to that of the (normally) incident light for a plate 0.001 mm. thick, for greenish yellow light ($\lambda_0 = 0.0005458$ mm.), and for different refractive indices (ordinates) and absorption indices (abscissæ). The pronounced effect of even a small absorption index in cutting down the transmission is evident from the curves.

A series of values computed by means of this equation on the basis of thickness $l = 0.001$ mm. (1 micron about $\frac{1}{20}$ the thickness of an ordinary rock thin-section) are listed in Table I. and represented

graphically in Fig. 1. From these the pronounced effect of κ in cutting out and absorbing the transmitted light energy is clearly evident. For any given thickness l , refractive index n , and absorption index κ , the amount of light absorbed can be obtained directly. From Fig. 1 it is evident that for a plate only 1 micron thick and for $\kappa > 0.5$ the amount of transmitted light is almost negligible and the plate is practically opaque. The effect of using a plate $\frac{1}{10}$ as thick is the same as that produced by a plate of the original thickness but with an absorption index $\frac{1}{10}$ as large. Thus a plate of refractive index $n=2$, thickness $l=0.001$ mm. and absorption index $\kappa=0.5$ transmits 10^{-10} of the incident light while a plate of the same refractive and absorption indices but $\frac{1}{10}$ as thick (0.0001 mm.) transmits 10^{-1} of the incident light. If we assume an intensity of illumination of 1,000 meter-candles and a threshold limit of vision 0.001 meter-candles, then 10^{-6} is the least amount of light which can be detected; under these conditions for an ordinary rock section of thickness 0.02 mm., equation (2b) reduces to $n\kappa=0.03$; therefore a mineral in a thin rock section whose absorption index κ exceeds 0.02, will appear perfectly opaque. The influence of even a very small absorption index is therefore exceedingly great in absorbing light energy.

TABLE 1.

In this table are listed the absorption indices κ of absorbing crystal plates of thickness $l=0.001$ mm. and refractive index n which transmit given quantities I/I_0 of the incident light. Thus a plate of this thickness, of refractive index $n=2.0$ and absorption index $\kappa=0.030$ transmits 0.25 of the incident light.

I/I_0	0.8	0.67	0.50	0.25	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
n														
1.0	.010	.018	.030	.060	.100	.200	.300	.400	.500	.600	.700	.800	.900	1.000
2.0	.005	.009	.015	.030	.050	.100	.150	.200	.250	.300	.350	.400	.450	.500
3.0	.003	.006	.010	.020	.033	.067	.100	.133	.167	.200	.233	.267	.300	.333
4.0	.002	.004	.008	.015	.025	.050	.075	.100	.125	.150	.175	.200	.225	.250
5.0	.002	.004	.006	.012	.020	.040	.060	.080	.100	.120	.140	.160	.180	.200
7.5	.001	.002	.004	.008	.013	.027	.040	.053	.067	.080	.093	.107	.120	.133
10.0	.001	.002	.003	.006	.010	.020	.030	.040	.050	.060	.070	.080	.090	.100

The characteristic feature of waves traversing an absorbing medium is the decrease in their vibration amplitude with distance of penetration; the waves are damped; the result is that for obliquely incident waves the vibration amplitudes along each wave front are not constant. In other words, the

surfaces of equal phase do not coincide with the surfaces of equal amplitude as in transparent media. In the electromagnetic theory transparent media are grouped under dielectrics (electric non-conductors) and in these Maxwell showed the possibility of the development of an electric current under the influence of an electric force, the current being proportional to the rate of change of the electric force; the current results on the shift in the density of the lines of electric force and varies in intensity with the rate of their displacement. If E is the electric force, the "displacement" current is proportional to $\partial E/\partial t$ and is in fact equal to $\epsilon \cdot \partial E/4\pi \cdot \partial t$ where ϵ is the dielectric constant. A current is surrounded by a magnetic field such that the lines of magnetic force are closed curves. Maxwell showed that the work done in carrying an isolated magnetic pole around one of these curves is equal to 4π times the electric current measured in electromagnetic units. Similarly a magnetic current or flux varies when the strength of the magnetic field changes and the lines of flow are surrounded by an electric field such that the lines of electric force are closed curves; the line integral of the electric force around one of these curves is numerically equal to 4π times the magnetic flux; furthermore the magnetic flux is proportional to the rate of change of the magnetic force or $\partial M/\partial t$ and is in fact equal to $\mu \cdot \partial M/4\pi \cdot \partial t$, in which μ is the magnetic permeability. For vibrations of the high frequency of light waves μ is practically equal to unity. From these relations Maxwell deduced the following differential equations:

$$\frac{\epsilon}{c} \frac{\partial X}{\partial t} = \frac{\partial v}{\partial y} - \frac{\partial v}{\partial z}, \quad \frac{\epsilon}{c} \frac{\partial Y}{\partial t} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \frac{\epsilon}{c} \frac{\partial Z}{\partial t} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad (3a)$$

$$\frac{1}{c} \frac{\partial u}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \quad \frac{1}{c} \frac{\partial v}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \quad \frac{1}{c} \frac{\partial w}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}, \quad (3b)$$

in which X, Y, Z and u, v, w are components of the electric and magnetic forces respectively and c is the ratio of the electrostatic to the electromagnetic systems of units and is numerically equal to the velocity of light.

To account for the phenomena observed in absorbing media the above equations are modified according to one of two possible hypotheses both of which involve the movements of electrons; the movements thus set up (translation in conductors, vibration in selective absorbers) absorb or divert part of the electromagnetic energy and convert it into heat (ohmic heating) and reduce the amount of energy available for the light vibrations. In electric conductors an electric force sets in motion streams of negatively charged electrons which are in effect the conduction current. The reaction is directly proportional to the impelling electric force, E , or to $\sigma \cdot E$ in which σ is the absolute conductivity in electrostatic units (Ohm's law). A medium which is not a perfect insulator exhibits properties intermediate between those of a dielectric and of a conductor and the current consists of two parts: a displacement current and a conduction current. The expression for the current which covers both cases is then

$$\frac{\epsilon}{4\pi} \cdot \frac{\partial E}{\partial t} + \sigma \cdot E. \quad (4)$$

In light-waves the disturbance is periodic in effect and the time occurs only in the form of a factor the most general expression for which is $\bar{A}e^{i\frac{2\pi}{T}t}$; in this factor T is the period of vibration and $\bar{A}$, which may be real, imaginary, or complex, is the amplitude vector. Under these conditions the expression (4) for the current may be written

$$\frac{1}{4\pi} (\epsilon - 2T\sigma i) \frac{\partial E}{\partial t}.$$

The only difference between this expression for the current and that for dielectrics is the replacement of the dielectric constant ϵ by the complex quantity $(\epsilon - 2T\sigma i)$, in which $i = \sqrt{-1}$.

In crystals the dielectric constant and the conductivity are different in different directions. Experience has shown that the components of the electric displacement for any direction of wave propagation are homogeneous, linear functions of the field components. Thus the X, Y, Z components of the conduction current are

$$\sigma_{11}X + \sigma_{12}Y + \sigma_{13}Z,$$

$$\sigma_{21}X + \sigma_{22}Y + \sigma_{23}Z,$$

$$\sigma_{31}X + \sigma_{32}Y + \sigma_{33}Z,$$

those of the displacement current

$$\frac{1}{4\pi} \left(\epsilon_{11} \frac{\partial X}{\partial t} + \epsilon_{12} \frac{\partial Y}{\partial t} + \epsilon_{13} \frac{\partial Z}{\partial t} \right), \text{ etc.,}$$

wherein $\sigma_{hk} = \sigma_{kh}$ and $\epsilon_{hk} = \epsilon_{kh}$. The first Maxwell equation becomes for absorbing crystals

$$\frac{1}{c} \left[(\epsilon_{11} - 2T\sigma_{11}i) \frac{\partial X}{\partial t} + (\epsilon_{12} - 2T\sigma_{12}i) \frac{\partial Y}{\partial t} + (\epsilon_{13} - 2T\sigma_{13}i) \frac{\partial Z}{\partial t} \right] = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},$$

or if $\bar{\epsilon}_{hk}$ be substituted for the complex expression $(\epsilon_{hk} - 2T\sigma_{hk}i)$ the Maxwell equations can be written

$$\left. \begin{aligned} \frac{1}{c} \left(\bar{\epsilon}_{11} \frac{\partial X}{\partial t} + \bar{\epsilon}_{12} \frac{\partial Y}{\partial t} + \bar{\epsilon}_{13} \frac{\partial Z}{\partial t} \right) &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \xi, \\ \frac{1}{c} \left(\bar{\epsilon}_{21} \frac{\partial X}{\partial t} + \bar{\epsilon}_{22} \frac{\partial Y}{\partial t} + \bar{\epsilon}_{23} \frac{\partial Z}{\partial t} \right) &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \eta, \\ \frac{1}{c} \left(\bar{\epsilon}_{31} \frac{\partial X}{\partial t} + \bar{\epsilon}_{32} \frac{\partial Y}{\partial t} + \bar{\epsilon}_{33} \frac{\partial Z}{\partial t} \right) &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \zeta, \end{aligned} \right\} \quad (5a)$$

$$\frac{1}{c} \frac{\partial u}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \quad \frac{1}{c} \frac{\partial v}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \quad \frac{1}{c} \frac{\partial w}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}, \quad (5b)$$

wherein $\bar{\epsilon}_{hk} = \bar{\epsilon}_{kh}$.

If for abbreviation the right hand side of equations (5a) be made equal respectively to ξ, η, ζ , the differential $\partial X/\partial t, \partial Y/\partial t, \partial Z/\partial t$ can be expressed

in the form

$$\left. \begin{aligned} \frac{\partial X}{\partial t} &= \frac{1}{c} (\bar{a}_{11}\xi + \bar{a}_{12}\eta + \bar{a}_{13}\zeta), \\ \frac{\partial Y}{\partial t} &= \frac{1}{c} (\bar{a}_{21}\xi + \bar{a}_{22}\eta + \bar{a}_{23}\zeta), \\ \frac{\partial Z}{\partial t} &= \frac{1}{c} (\bar{a}_{31}\xi + \bar{a}_{32}\eta + \bar{a}_{33}\zeta), \end{aligned} \right\} \quad (6)$$

in which the complex constants (called polarization constants) $\bar{a}_{h\kappa} = a_{h\kappa} + i b_{h\kappa}$ are simple determinate functions of c^2 and of the complex quantities $\bar{e}_{11} \dots \bar{e}_{22} \dots$; in these equations $\bar{a}_{h\kappa} = \bar{a}_{\kappa h}$.

To eliminate the components of the electric force X, Y, Z differentiate equations (5b) after the time

$$\begin{aligned} \frac{1}{c} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial z} \left(\frac{\partial Y}{\partial t} \right) - \frac{\partial}{\partial y} \left(\frac{\partial Z}{\partial t} \right), & \frac{1}{c} \frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial x} \left(\frac{\partial Z}{\partial t} \right) - \frac{\partial}{\partial z} \left(\frac{\partial X}{\partial t} \right), \\ \frac{1}{c} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial y} \left(\frac{\partial X}{\partial t} \right) - \frac{\partial}{\partial y} \left(\frac{\partial Y}{\partial t} \right), \end{aligned} \quad (7)$$

and substitute in these equations the values of $\partial X/\partial t, \partial Y/\partial t, \partial Z/\partial t$ from equation (6), and obtain the equations

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x} (\bar{a}_{21}\xi + \bar{a}_{22}\eta + \bar{a}_{23}\zeta) - \frac{\partial}{\partial y} (\bar{a}_{31}\xi + \bar{a}_{32}\eta + \bar{a}_{33}\zeta), \\ \frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial x} (\bar{a}_{31}\xi + \bar{a}_{32}\eta + \bar{a}_{33}\zeta) - \frac{\partial}{\partial z} (\bar{a}_{11}\xi + \bar{a}_{12}\eta + \bar{a}_{13}\zeta), \\ \frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial y} (\bar{a}_{11}\xi + \bar{a}_{12}\eta + \bar{a}_{13}\zeta) - \frac{\partial}{\partial x} (\bar{a}_{21}\xi + \bar{a}_{22}\eta + \bar{a}_{23}\zeta), \end{aligned} \right\} \quad (8)$$

which are free from the components of the electric force and of the electric current.

A particular solution of these equations is

$$\begin{aligned} u &= \mu_1 p, & v &= \mu_2 p, & w &= \mu_3 p, \\ p &= \bar{A} e^{i \frac{2\pi}{T}} \left(t - \frac{\nu_1 x + \nu_2 y + \nu_3 z}{\bar{q}} \right), \end{aligned} \quad (9)$$

in which μ_1, μ_2, μ_3 are the direction cosines and $\bar{A}$ (complex) the amplitude of the polarization vector; ν_1, ν_2, ν_3 the direction cosines of the wave normal and $\bar{q}$ (complex) the velocity. If in (9) we assume vertical incidence, then

$\nu_1 = \nu_2 = 0, \nu_3 = 1$; if we put $\frac{1}{\bar{q}} = \frac{1 - i\kappa}{q}$, and $\lambda = T \cdot q$, wherein q, κ , and λ are real, then

$$u = \mu_1 \bar{A} e^{-2\pi\kappa \cdot \frac{z}{\lambda}} \cdot e^{i2\pi \left(\frac{t}{T} - \frac{z}{\lambda} \right)}. \quad (10)$$

This equation states that the amplitude of the wave of light after passage through the path λ (one wave length) has decreased to the extent of $e^{-2\pi\kappa}$; κ is therefore called the absorption index.

For a beam of light entering a second medium the axes of reference may be so chosen that the Z -axis is normal to the boundary surface while the plane of incidence is the XZ plane. In this case $\nu_1 = \sin i$, $\nu_2 = 0$, $\nu_3 = \cos i$, if the positive Z -axis points into the second medium and i is the angle of incidence. In adopting this convention and substituting the values of (9) in (8 and 5a) we obtain, on eliminating μ_1 , μ_2 , μ_3 the equation

$$\left. \begin{aligned} (\bar{q}^2 - a_{22})(\bar{q}^2 - \bar{a}_{11} \cos^2 i - \bar{a}_{33} \sin^2 i + 2\bar{a}_{31} \sin i \cos i) \\ = (\bar{a}_{23} \sin i - \bar{a}_{12} \cos i)^2. \end{aligned} \right\} \quad (11)$$

In this equation $\bar{q}$ and the constants $\bar{a}_{11} \dots$ are complex quantities. In similar manner we find, by observing that the wave normal and the polarization vector are at right angles, the expression

$$\mu_1(\bar{q}^2 - \bar{a}_{22}) = \mu_2(\bar{a}_{23} \cos i \sin i - \bar{a}_{12} \cos^2 i). \quad (12)$$

If the polarization plane include an angle $\bar{\delta}$ with the plane of incidence ($\mu_1 = -\cos i \cos \bar{\delta}$, $\mu_2 = \sin \bar{\delta}$, $\mu_3 = \sin i \cos \bar{\delta}$) equation (12) can be written

$$\tan \bar{\delta} = \frac{\bar{q}^2 - \bar{a}_{22}}{\bar{a}_{12} \cos i - \bar{a}_{23} \sin i} = \frac{\bar{a}_{12} \cos i - \bar{a}_{23} \sin i}{\bar{q}^2 - \bar{a}_{11} \cos^2 i - \bar{a}_{33} \sin^2 i + 2\bar{a}_{31} \sin i \cos i}. \quad (12a)$$

By means of this equation the azimuth of the wave $\bar{q}_k$ can be computed.

Boundary Conditions.—On passing from one medium to another, as from air into a crystal plate, light waves encounter at the boundary surface of the plate entirely new conditions. This passage from the system of forces operative in the first medium to that in the second is exceedingly rapid and is accomplished within a very thin film; but it is nevertheless a continuous process since, physically speaking, there are no discontinuities in nature. The boundary surface is an inhomogeneous film in which the dielectric constant passes continuously, though rapidly, from that of the first to that of the second medium. Now in order that the finite current be carried across the boundary the components of the electric and magnetic forces parallel with the boundary surface must be continuous through the boundary; for an infinitely thin film the forces on either side of the film must therefore be equal. If the boundary surface be the xy plane and the plane of incidence the xz plane, the general boundary conditions are $(u)_1 = (u)_2$, $(v)_1 = (v)_2$, $(X)_1 = (X)_2$, $(Y)_1 = (Y)_2$. For periodic vibrations it is evident that $(\partial X / \partial t)_1 = (\partial X / \partial t)_2$ and $(\partial Y / \partial t)_1 = (\partial Y / \partial t)_2$ may be used in place of $(X)_1 = (X)_2$, $(Y)_1 = (Y)_2$; the last equation of (5b) can be written $(\partial w / \partial t)_1 = (\partial w / \partial t)_2$, or $(w)_1 = (w)_2$ for periodic vibrations. Only four of these conditions are independent.

In the case of light waves entering a crystal plate from air, there are two components (magnetic vectors) on the air side of the boundary film, u_E , that of the incident wave and u_R that of the reflected wave; on the crystal side there are the components u_1 , u_2 of the two refracted waves. Equations (8) define the state of the light wave motion at any point and instant. For the boundary film $Z = 0$; in order that in the film the boundary conditions hold for all instants of time, the relations must obtain

$$\frac{\nu_1}{q_1} = \frac{\nu_2}{q_2} = \frac{1}{f} \quad \text{or} \quad \frac{\sin i}{q_0} = \frac{\sin \bar{r}}{\bar{q}} = \frac{1}{f}$$

in which i is the angle of incidence, $\bar{r}$ angle¹ of refraction, $\bar{q}_0$ velocity of light in air, q wave normal velocity in the crystal medium and f a constant, real number. Since $\bar{q}$ is complex, $\bar{r}$ is also complex. By means of these values equations 11 and 12a can be transformed to read

$$[\bar{a}_{22} + (\bar{a}_{22} - f^2) \tan^2 \bar{r}] [\bar{a}_{11} - 2\bar{a}_{31} \tan \bar{r} + (\bar{a}_{33} - f^2) \tan^2 \bar{r}] \\ = (\bar{a}_{12} - \bar{a}_{23} \tan^2 \bar{r})(1 + \tan^2 \bar{r}) \quad (13)$$

$$\tan \bar{\delta} = \frac{(f^2 - \bar{a}_{22}) \tan^2 \bar{r} - \bar{a}_{22}}{(\bar{a}_{12} - \bar{a}_{23} \tan^2 \bar{r}) \sqrt{1 + \tan^2 \bar{r}}} = \frac{(f^2 - \bar{a}_{22}) \tan^2 \bar{r} - \bar{a}_{22}}{(f^2 - \bar{a}_{33}) \tan^2 \bar{r} + 2\bar{a}_{31} \tan \bar{r} - \bar{a}_{11}}. \quad (14)$$

In these equations $\bar{a}_{11}, \dots, \bar{r}$ and $\bar{\delta}$ are complex quantities. A complex value of $\bar{r}$ signifies that the amplitude in the wave is not constant; a complex value of $\bar{\delta}$, that there is a phase difference between the amplitudes of the components normal and parallel to the plane of incidence, hence the vibration is elliptic in form.

For the case assumed, namely, a crystal plate surrounded by air, the boundary conditions $(u)_1 = (u)_0$, $(v)_1 = (v)_2$, $(w)_1 = (w)_2$, $(\partial X / \partial t)_1 = (\partial X / \partial t)_2$ can be written by virtue of equations (9), (5a), (6), and (7).

$$\left. \begin{aligned} (E \cos \epsilon - \bar{R} \cos \bar{\rho}) \cos i &= \bar{D}_1 \cos \bar{\delta}_1 \cos \bar{r}_1 + \bar{D}_2 \cos \bar{\delta}_2 \cos \bar{r}_2, \\ E \sin \epsilon + \bar{R} \sin \bar{\rho} &= \bar{D}_1 \sin \bar{\delta}_1 + \bar{D}_2 \sin \bar{\delta}_2, \\ (E \cos \epsilon + \bar{R} \cos \bar{\delta}) \sin i &= \bar{D}_1 \cos \bar{\delta}_1 \sin \bar{r}_1 + \bar{D}_2 \cos \bar{\delta}_2 \sin \bar{r}_2, \\ (E \sin \epsilon - \bar{R} \sin \bar{\rho}) \sin i \cdot \cos i & \\ &= \bar{D}_1 \frac{\sin \bar{r}_1}{\bar{q}_1^2} [\sin \bar{\delta}_1 (\bar{a}_{11} \cos \bar{r}_1 - \bar{a}_{13} \sin \bar{r}_1) + \bar{a}_{12} \cos \bar{\delta}_1] \\ &+ \bar{D}_2 \frac{\sin \bar{r}_2}{q_2^2} [\sin \bar{\delta}_2 (\bar{a}_{11} \cos \bar{r}_2 - \bar{a}_{13} \sin \bar{r}_2) + \bar{a}_{12} \cos \bar{\delta}_2], \end{aligned} \right\} \quad (15)$$

in which $E, \bar{R}, \bar{D}_1, \bar{D}_2$ are the amplitudes of the incident, reflected, and two refracted waves respectively; $\epsilon, \bar{\rho}, \bar{\delta}, \bar{\delta}_2$, the polarization azimuths; $q_0, q_0, \bar{q}_1, \bar{q}_2$, the normal velocities of the waves, $i, -i, \bar{r}_1, \bar{r}_2$, the angles between the wave normals and the plate normal.

The last equation of (15) may also be written

$$(E \sin \epsilon - \bar{R} \sin \bar{\rho}) \sin i \cos i = \bar{D}_1 \sin \bar{r}_1 (\cos \bar{r}_1 \sin \bar{\delta} \pm \sin \bar{r}_1 \tan \bar{s}_1) \\ + \bar{D}_2 \sin \bar{r}_2 (\cos \bar{r}_2 \sin \bar{\delta}_2 \pm \sin \bar{r}_2 \tan \bar{s}_2), \quad (15e)$$

wherein $\bar{s}_h$ is the angle between the refracted wave normal and its ray direction. In this form the equation is more convenient for use in certain computations. In these equations i, E, ϵ, q_0 of the incident wave are known; i, q_0 of the reflected wave; also by computation from (12) and (13) $\bar{r}_1, \bar{r}_2$ and $\bar{\delta}_1, \bar{\delta}_2$ of the refracted waves; unknown are $\bar{R}, \bar{\rho}$ of the reflected wave and $\bar{D}_1, \bar{D}_2$ of the refracted waves.

¹ A dash above a letter is used to signify that the quantity represented may be complex.

The computation of these four quantities from the equations is exceedingly complicated. The introduction of "uniradial azimuths" facilitates the solution considerably. Thus for a certain value of ϵ such as ϵ_1 , all of the refracted light takes the path $\bar{D}$ ($\bar{D}_1=1, \bar{D}_2=0$), while for another azimuth of the incident plane polarized wave, ($\bar{D}_2=1, \bar{D}_1=0$) and all of the light is refracted along $\bar{D}_2$. The values of the uniradial azimuths are given by the equations

$$\left. \begin{aligned} \tan \rho &= -\cos(i + \bar{r}) \tan \bar{\delta} \pm \frac{\sin^2 \bar{r} \cdot \tan \bar{s}}{\sin(i - \bar{r}) \cos \bar{\delta}}, \\ \tan \epsilon &= \cos(i - \bar{r}) \tan \bar{\delta} \pm \frac{\sin^2 \bar{r} \cdot \tan \bar{s}}{\sin(i + \bar{r}) \cos \bar{\delta}}. \end{aligned} \right\} \quad (16)$$

in which $\bar{s}$ is the angle between the wave normal and the ray direction of the refracted wave.

Let the values of E and $\bar{R}$, computed under the assumption that $\bar{D}_2=0, \bar{D}_1=1$, be E_1, R_1 ; similarly for $\bar{D}_1=0, \bar{D}_2=1$, let the values be E_2, R_2 . Since equations (15) and (15 ϵ) are linear in $\bar{D}_1$ and $\bar{D}_2$, we find on substituting therein first $\bar{D}_2=0, \bar{D}_1=1$ and then $\bar{D}_1=0, \bar{D}_2=1$, and indicating by subscripts, 1 or 2, in each case the proper uniradial azimuth, multiplying the equations thus obtained by $\bar{D}_1$ and $\bar{D}_2$ respectively and then adding

$$[(\bar{E}_1 \cos \epsilon_1 \bar{D}_1 + E_2 \cos \epsilon_2 \bar{D}_2) - (\bar{R}_1 \cos \delta_1 \bar{D}_1 + \bar{R}_2 \cos \delta_2 \bar{D}_2)] \cos i \\ = \bar{D}_1 \cos \delta_1 \cos \bar{r}_1 + \bar{D}_2 \cos \delta_2 \cos \bar{r}_2;$$

therefore

$$\left. \begin{aligned} E \cos \epsilon &= E p = E_1 \cos \epsilon_1 \bar{D}_1 + E_2 \cos \epsilon_2 \bar{D}_2 = E p_1 \bar{D}_1 + E p_2 \bar{D}_2, \\ \bar{R} \cos \rho &= \bar{R} p = \bar{R}_1 \cos \rho_1 \bar{D}_1 + \bar{R}_2 \cos \rho_2 \bar{D}_2 = \bar{R} p_1 \bar{D}_1 + \bar{R} p_2 \bar{D}_2, \\ E \sin \epsilon &= E s = E_1 \sin \epsilon_1 \bar{D}_1 + E_2 \sin \epsilon_2 \bar{D}_2 = E s_1 \bar{D}_1 + E s_2 \bar{D}_2, \\ \bar{R} \sin \rho &= \bar{R} s = \bar{R}_1 \sin \rho_1 \bar{D}_1 + \bar{R}_2 \sin \rho_2 \bar{D}_2 = \bar{R} s_1 \bar{D}_1 + \bar{R} s_2 \bar{D}_2, \end{aligned} \right\} \quad (17)$$

in which

$$E s_h = E_h \sin \epsilon_h, E p_h = E_h \cos \epsilon_h; \bar{R} s_h = \bar{R}_h \sin \epsilon_h, \bar{R} p_h = \bar{R}_h \cos \epsilon_h.$$

If the amplitude and polarization azimuth of the incident wave be given and hence $E s_1, E p_1, \delta_h$ and r_h , the corresponding values of the reflected wave can be computed from equations (17) and put into the form

$$\left. \begin{aligned} (E p_1 E s_2 - E p_2 E s_1) \bar{R} s_1 &= (\bar{R} s_1 E s_2 - \bar{R} s_2 E s_1) E p - (\bar{R} s_1 E p_2 - \bar{R} s_2 E p_1) E s, \\ (E p_1 E s_2 - E p_2 E s_1) \bar{R} p &= (\bar{R} p_1 E s_2 - \bar{R} p_2 E s_1) E p - (\bar{R} p_1 E p_2 - \bar{R} p_2 E p_1) E s, \end{aligned} \right\} \quad (18)$$

which is deduced by eliminating $\bar{D}_1$ and D_2 from equations (17). The values of $\bar{R}_s$ and $\bar{R}_p$ are complex.

If the incident wave be plane-polarized the complex values of $\bar{R}_s/E_s, \bar{R}_p/E_p$, and $\bar{R}_s/\bar{R}_p$ show that phase differences exist between the three waves and that therefore the reflected wave is in general elliptically polarized. If

we put

$$\frac{\bar{R}s}{\bar{R}p} = \tan \rho' = \tan \psi e^{i\Delta} = \tan \psi (\cos \Delta + i \sin \Delta) \quad (19)$$

$$= \frac{(\bar{R}s_1 E s_2 - \bar{R}s_2 E s_1) - (\bar{R}s_1 E p_2 - \bar{R}s_2 E p_1) \tan \epsilon}{(\bar{R}p_1 E s_2 - \bar{R}p_2 E s_1) - (\bar{R}p_1 E p_2 - \bar{R}p_2 E p_1) \tan \epsilon},$$

in which $\tan \epsilon = Es/Ep$ and the azimuth angle ψ is real, then Δ is the relative phase difference of the components normal and parallel to the plane of incidence. By equating the real and imaginary parts of this equation to zero, we obtain equations from which ψ and Δ can be computed. The equation shows that for a certain angle of incidence the phase difference may be $\pi/2$; for this angle of incidence the reflected wave is plane-polarized. If $\epsilon = 45^\circ$, then this angle of incidence, i , is called the principal angle of incidence and the angle Ψ the principal azimuth. From these two angles it is possible in certain cases (especially isotropic bodies) to compute the refractive index and the absorption index of the reflecting medium. Different methods have been devised for ascertaining these angles; practically all the methods available for ascertaining the refractive and absorption indices of absorbing substances are based on the above equations or certain modifications of the same, deduced on the basis of simplifying assumptions.

VERTICAL INCIDENCE.

In case the angle of incidence is zero, $i=0$, the above equations reduce to the following:

$$(\bar{q}^2 - \bar{a}_{22})(\bar{q}^2 - \bar{a}_{11}) = \bar{a}_{12}^2, \quad (11')$$

$$\tan \bar{\delta} = \frac{\bar{q}^2 - \bar{a}_{22}}{\bar{a}_{12}} = \frac{\bar{a}_{12}}{\bar{q}^2 - \bar{a}_{11}}, \quad (12a')$$

$$E \cos \epsilon - R \cos \rho = D_1 \cos \bar{\delta}_1 + D_2 \cos \bar{\delta}_2,$$

$$E \sin \epsilon + R \sin \rho = D_1 \sin \bar{\delta}_1 + D_2 \sin \bar{\delta}_2, \quad (15', 15'e)$$

$$q_0 E \cos \epsilon + q_0 R \cos \rho = \bar{q}_1 D_1 \cos \bar{\delta}_1 + \bar{q}_2 D_2 \cos \bar{\delta}_2,$$

$$q_0 E \sin \epsilon - q_0 R \sin \rho = \bar{q}_1 D_1 \sin \bar{\delta}_1 + \bar{q}_2 D_2 \sin \bar{\delta}_2.$$

For uniradial azimuths we find for δ_1 ,

$$Ep_1 - Rp_1 = \cos \bar{\delta}_1,$$

$$Es_1 + Rs_1 = \sin \bar{\delta}_1,$$

$$Ep_1 + Rp_1 = \frac{\bar{q}_1}{q_0} \cos \bar{\delta}_1,$$

$$Es_1 - Rs_1 = \frac{\bar{q}_1}{q_0} \sin \bar{\delta}_1.$$

Therefore

$$2q_0 E p_1 = (q_0 + \bar{q}_1) \cos \bar{\delta}_1, \quad 2q_0 E s_1 = (q_0 + \bar{q}_1) \sin \bar{\delta}_1; \\ 2q_0 R p_1 = (\bar{q}_1 - q_0) \cos \delta_1, \quad 2q_0 \bar{R} s_1 = (q_0 - \bar{q}_1) \sin \bar{\delta}_1.$$

Similarly

$$2q_0 E p_2 = (q_0 + \bar{q}_2) \cos \bar{\delta}_2, \quad 2q_0 E s_2 = (q_0 + \bar{q}_2) \sin \bar{\delta}_2; \\ 2q_0 \bar{R} p_2 = (\bar{q}_2 - q_0) \cos \bar{\delta}_2, \quad 2q_0 \bar{R} s_2 = (q_0 - \bar{q}_2) \sin \bar{\delta}_2.$$

On substituting these values in equation (19), we obtain

$$\frac{\bar{R}s}{\bar{R}p} = \frac{2q_0(\bar{q}_2 - \bar{q}_1) \tan \bar{\delta}_1 \tan \bar{\delta}_2 - [(q_0 - \bar{q}_1)(q_0 + \bar{q}_2) \tan \bar{\delta}_1 \\ - (q_0 - \bar{q}_2)(q_0 + \bar{q}_1) \tan \bar{\delta}_2] \tan \epsilon}{(q_0 - \bar{q}_2)(q_0 + \bar{q}_1) \tan \bar{\delta}_1 - (q_0 - \bar{q}_1)(q_0 + \bar{q}_2) \tan \bar{\delta}_2 \\ - 2q_0(q_2 - \bar{q}_1) \tan \epsilon}. \quad (19')$$

The right-hand side of this equation is complex; therefore, plane polarized light incident on a crystal plate of an absorbing crystal generally becomes on reflection elliptically polarized. The equations are, however, so complicated, that progress is best made by the solution of a few simple cases in which the crystallographic symmetry relations prescribe certain types of vibration.

For the special case that the plane of incidence is a plane of symmetry the equations become noticeably simpler; this assumption is valid for isotropic, uniaxial, and certain sections of orthorhombic crystals because in these the positions of the principal axes of the polarization and absorption surfaces of reference coincide as they are fixed by the symmetry relations. In general these axes do not coincide and the surface of reference can be represented only by the use of complex quantities. On the assumption that the plane of incidence is a plane of symmetry let θ be the angle between the Z' axis (normal to the plate) and the Z principal axis. The equations defining the complex polarization constants then obtain

$$\bar{a}_{11} = a_{11} + ib_{11} = \bar{a}^2 \cos^2 \theta + \bar{c}^2 \sin^2 \theta,$$

$$\bar{a}_{22} = a_{22} + ib_{22} = \bar{b}^2,$$

$$\bar{a}_{33} = a_{33} + ib_{33} = \bar{a}^2 \sin^2 \theta + \bar{c}^2 \cos^2 \theta,$$

$$\bar{a}_{23} = a_{23} + ib_{23} = 0,$$

$$\bar{a}_{31} = a_{31} + ib_{31} = (\bar{a}^2 - \bar{c}^2) \cos \theta \sin \theta,$$

$$\bar{a}_{12} = a_{12} + ib_{12} = 0.$$

In these equations $\bar{a}_{11}$, $\dots$ and $\bar{a}$, b , $\bar{c}$ are complex quantities. On substituting these values in equations (10'), (11'a) and (15) we find

$$(q^2 - \bar{a}_{11})(q^2 - \bar{a}_{22}) = 0 \quad \text{or} \quad \bar{q}_2^2 = \bar{a}_{11} \quad \bar{q}_1^2 = \bar{a}_{22},$$

$$\delta_1 = 0, \quad \delta_2 = \frac{\pi}{2},$$

$$\epsilon_1 = 0, \quad \epsilon_2 = \frac{\pi}{2}.$$

The refracted waves δ_1 , δ_2 are thus plane-polarized. Equations (18) reduce to the form

$$\frac{\bar{R}s}{\bar{E}s} = \frac{Rs_1 E p_2}{E p_2 E s_1} = \frac{Rs_1}{E s_1} = \frac{q_0 - \bar{q}_1}{q_0 + \bar{q}_1}.$$

But from equation (10)

$$\frac{\bar{q}_1}{q_0} = \frac{q_1}{q_0(1 - i\kappa)} = \frac{1}{n_1(1 - i\kappa)};$$

accordingly

$$\frac{\bar{R}s}{\bar{E}s} = \frac{n_1 - 1 - in_1\kappa_1}{n_1 + 1 - in_1\kappa_1}. \quad (20)$$

The right-hand side of this equation is a complex quantity and of the form

$$\frac{a + ib}{A + iB} = \frac{aA + bB + i(bA - aB)}{A^2 + B^2} = f + ig.$$

This in turn may be considered equivalent to the expression

$$(P + iQ)e^{i\tau} = (P + iQ)(\cos \tau + i \sin \tau) \\ = P \cos \tau - Q \sin \tau + i(P \sin \tau + Q \cos \tau).$$

If now

$$P = r \cos \Delta,$$

$$Q = -r \sin \Delta,$$

the amplitude may be written for $E s = 1$

$$R s = r_1 \cos(\tau - \Delta_1) + i r_1 \sin(\tau - \Delta_1) = f + ig = r_1 e^{i(\tau - \Delta_1)} \quad (21)$$

Therefore

$$f = r_1 \cos(\tau - \Delta_1),$$

$$g = r_1 \sin(\tau - \Delta_1).$$

Hence

$$r_1^2 = f^2 + g^2$$

$$\tan(\tau - \Delta_1) = \frac{g}{f}$$

The expression $f^2 + g^2$ is most readily obtained by multiplying $f + ig$ by its complex conjugate $(f - ig)$. Accordingly the intensity of the reflected light which is proportional to the square of the amplitude r_1^2 is

$$\frac{Irs}{Is} = \frac{(n_1 - 1)^2 + n_1^2 \kappa_1^2}{(n_1 + 1)^2 + n_1^2 \kappa_1^2} = \frac{\left(1 - \frac{1}{n_1}\right)^2 + \kappa_1^2}{\left(1 + \frac{1}{n_1}\right)^2 + \kappa_1^2}. \quad (22)$$

The phase difference is given by

$$\tan(\tau - \Delta_1) = \frac{bA - aB}{aA + bB} = \frac{-2n_1\kappa_1}{n_1^2 - 1 + n_1\kappa_1}. \quad (23)$$

Similarly the amplitude of the component parallel with the plane of incidence is

$$\frac{Rp}{Ep} = \frac{Rp_2}{Ep_2} = \frac{\bar{q}_2 - q_0}{\bar{q}_2 + q_0} = -\frac{n_2 - 1 - in_2\kappa_2}{n_2 + 1 - in_2\kappa_2}. \quad (24)$$

The intensity is

$$\frac{Irp}{Ip} = \frac{(n_2 - 1)^2 + n_2^2 \kappa_2^2}{(n_2 + 1)^2 + n_2^2 \kappa_2^2}. \quad (25)$$

The phase difference is

$$\tan(\tau - \Delta_2) = \frac{-2n_2\kappa_2}{n_2^2 - 1 + n_2^2 \kappa_2^2}. \quad (26)$$

By division of (20) by (24) an expression for the amplitude ratio is obtained

$$\frac{Rs}{Rp} = -\frac{(n_1 - 1 - in_1\kappa_1)(n_2 + 1 - in_2\kappa_2)}{(n_1 + 1 - in_1\kappa_1)(n_2 - 1 - in_2\kappa_2)} \cdot \frac{Es}{Ep}. \quad (27)$$

This is a complex quantity and indicates that a phase difference exists between the two components (parallel and normal to plane of incidence) of the reflected light; the reflected light is in general elliptically polarized. In case the azimuth of the plane polarized incident light ϵ coincides with either ϵ_1 or ϵ_2 , the reflected light is plane-polarized. The azimuths for which the reflected light is plane-polarized occur at intervals of 90° .

The intensity ratio of the components of the reflected light is found by division of (21) by (25)

$$\frac{Irs}{Irp} = \frac{[(n_1 - 1)^2 + n_1^2 \kappa_1^2][(n_2 + 1)^2 + n_2^2 \kappa_2^2]}{[(n_1 + 1)^2 + n_1^2 \kappa_1^2][(n_2 - 1)^2 + n_2^2 \kappa_2^2]} \cdot \frac{Is}{Ip}. \quad (28)$$

The phase difference for $Ep = Es$ or $\tan \epsilon = 1$ is $\Delta_1 - \Delta_2$ or

$$\tan (\Delta_1 - \Delta_2) = \frac{2n_2 \kappa_2 (n_1^2 - 1 + n_1^2 \kappa_1^2) - 2n_1 \kappa_1 (n_2^2 - 1 + n_2^2 \kappa_2^2)}{(n_1^2 - 1 + n_1^2 \kappa_1^2)(n_2^2 - 1 + n_2^2 \kappa_2^2) + 4n_1 \kappa_1 n_2 \kappa_2}. \quad (29)$$

TRANSPARENT MEDIA.

In order to realize clearly the effects of absorption phenomena it is essential to ascertain first the behavior of non-absorbing media, then to pass to the more complex problem of absorbing bodies.

Isotropic Substances.—For non-absorbing bodies $\kappa = 0$; in this case equation (22) reduces to the ordinary Fresnel expression for the intensity of rays reflected from an isotropic plate

$$\frac{Ir}{I} = \left(\frac{n - 1}{n + 1} \right)^2. \quad (30)$$

TABLE 2.

In this table are listed the relative intensities of light reflected at vertical incidence from polished surfaces of substances of given refractive indices. Thus a plate of refractive index 2.1 reflects 0.1259 of the incident light.

n	$\frac{Ir}{I}$	n	$\frac{Ir}{I}$	n	$\frac{Ir}{I}$	n	$\frac{Ir}{I}$
1.0	0	1.6	.0533	2.2	.1406	2.8	.2244
1.1	.0023	1.7	.0676	2.3	.1552	2.9	.2374
1.2	.0083	1.8	.0816	2.4	.1696	3.0	.2500
1.3	.0170	1.9	.0963	2.5	.1836	4.0	.3600
1.4	.0278	2.0	.1111	2.6	.1975	5.0	.4444
1.5	.0400	2.1	.1259	2.7	.2111	10.0	.6667

This equation shows that the higher the refractive index the more light is reflected. The relative intensities for a series of refractive indices are listed in Table 2.

Equation (23) reduces for $\kappa=0$ to $\tan(\tau-\Delta_1)=0$; in other words, there is no change in phase on reflection. Vertically incident plane-polarized light is reflected as plane-polarized light without change of phase and with no change in azimuth of polarization plane. (Equation (27) for $n_1=n_2$ and $\kappa_1=\kappa_2=0$.)

Birefracting Media.—For a birefracting transparent medium ($\kappa_1=\kappa_2=0$); equation (28) reduces to

$$\frac{Irs}{Irp} = \left(\frac{n_1 - 1}{n_1 + 1} \right)^2 \left(\frac{n_2 + 1}{n_2 - 1} \right)^2$$

$$= 1 - \frac{4(n_2 - n_1)}{(n_1 + 1)(n_2 - 1)} + \frac{4(n_2 - n_1)^2}{(n_1 + 1)^2(n_2 - 1)^2} \quad (31)$$

The third term of the last expression is negligible for weakly birefracting substances. The change in intensity ratio with change in least refractive index n_1 and with birefringence ($n_2 - n_1$) is shown in Table 3 and presented graphically in Fig 2.

TABLE 3.

In this table are given the relative intensities of the two components, normal and parallel to the plane of incidence, of light waves reflected from a transparent birefracting crystal surface whose low refractive index is n_1 and whose birefringence is $n_2 - n_1$. Thus for a crystal plate, whose least refractive index is 1.8 and whose birefringence is 0.040, the ratio of the intensities of the two reflected components is 0.932.

$n_2 - n_1$	0.005	0.010	0.020	0.030	0.040	0.050	0.075	0.100	0.150	0.200
n_1										
1.2	.960	.919	.845	.794	.723	.673	.568	.487	.374	.298
1.4	.979	.960	.917	.888	.854	.824	.755	.695	.598	.522
1.6	.989	.975	.952	.928	.908	.886	.837	.793	.716	.652
1.8	.991	.982	.964	.948	.932	.916	.881	.848	.787	.734
2.0	.993	.986	.973	.962	.949	.937	.908	.882	.833	.789
2.5	.995	.991	.984	.977	.970	.963	.945	.929	.898	.869
3.0	.998	.995	.990	.986	.980	.976	.963	.953	.931	.911
4.0	.998	.996	.995	.992	.989	.986	.982	.974	.962	.951
5.0	.999	.998	.997	.995	.993	.992	.988	.985	.977	.968

The ratio of the amplitudes (equation 27) becomes

$$\frac{Rs}{Rp} = - \frac{(n_1 - 1)(n_2 + 1)}{(n_1 + 1)(n_2 - 1)} \cdot \frac{Es}{Ep}. \quad (32)$$

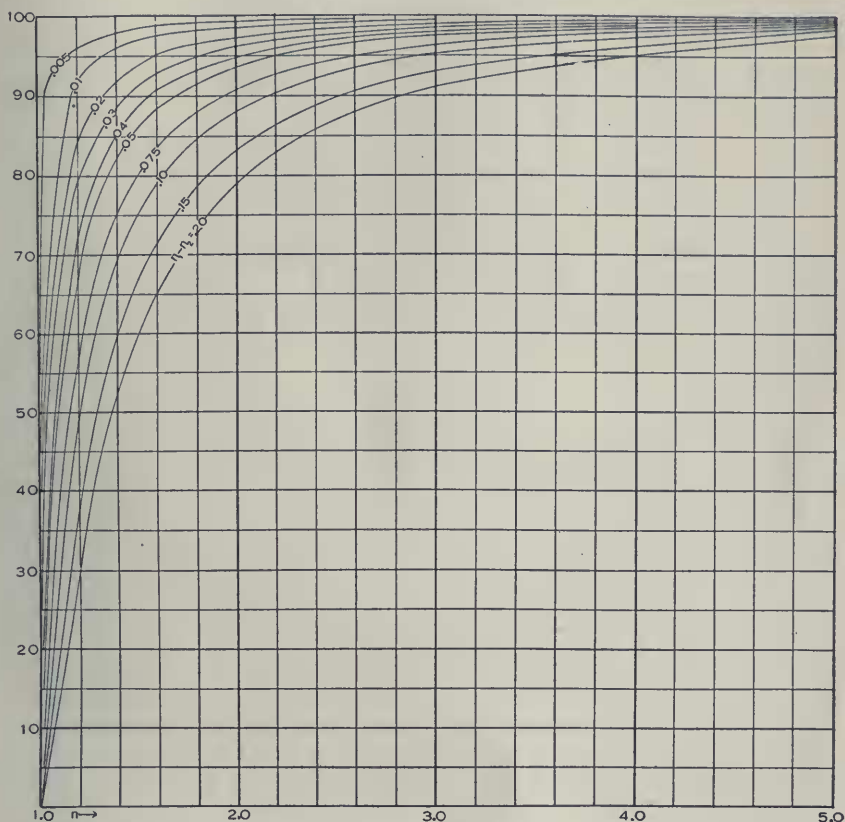


FIG. 2. The curves in this figure indicate the change in the percentage intensity ratio (ordinates) of the two components of normally reflected light with change in the least refractive index n_1 (abscissæ) and in the birefringence ($n_2 - n_1$) (curves) of a crystal plate.

The fact that this is a real quantity proves that there is no phase difference between the two components; the phenomena observed on such plates are due therefore entirely to a difference in the amplitudes of the reflected wave components. The reflected light is plane-

polarized; the azimuth ρ of the plane of polarization of the reflected wave is for $Es = E\rho$ or $\tan \epsilon = 1$ (equation 32)

$$\tan \rho = \frac{Rs}{R\rho} = -1 + \frac{2(n_2 - n_1)}{(n_1 + 1)(n_2 - 1)}. \quad (33)$$

There is therefore on reflection a rotation of the plane of polarization through the angle $(\epsilon - \rho)$. For the ordinary case $\tan \epsilon = 1$ ($\epsilon = 45^\circ$), we find from (33)

$$\tan (\epsilon - \rho) = \frac{1 - \tan \rho}{1 + \tan \rho} = \frac{n_1 n_2 - 1}{n_2 - n_1}. \quad (34)$$

The angular amounts of rotation on reflection of vertically incident plane-polarized light for $\epsilon = 45^\circ$ and for different refractive indices and birefringences are listed in Table 4 and are presented graphically in Fig. 3.

TABLE 4.

In this table is given the angular rotation, on reflection from a crystal plate, of the plane of polarization of vertically incident light waves whose plane of polarization includes an angle of 45° with the vibration directions of the crystal plate.

n_1	1.2	1.4	1.6	1.8	2.0	2.5	3.0	4.0	5.0
$n_2 - n_1$									
.005	0° 39'	0° 18'	0° 11'	0° 08'	0° 05'	0° 03'	0° 02'	0° 01'	0° 01'
.010	1 16	0 35	0 22	0 15	0 11	0 06	0 04	0 02	0 01
.020	2 28	1 09	0 43	0 30	0 22	0 13	0 08	0 05	0 03
.030	3 36	1 43	1 04	0 45	0 35	0 19	0 13	0 07	0 04
.040	4 41	2 15	1 25	0 59	0 44	0 26	0 17	0 09	0 06
.050	5 42	2 47	1 45	1 14	0 55	0 32	0 21	0 11	0 07
.075	8 03	4 02	2 33	1 48	1 22	0 48	0 31	0 17	0 10
.100	10 08	5 11	3 19	2 22	1 47	1 03	0 41	0 22	0 14
.150	13 36	7 18	4 46	3 26	2 36	1 32	1 01	0 33	0 21
.200	16 23	9 10	6 04	4 24	3 19	2 00	1 20	0 43	0 28

Figures 2 and 3 indicate clearly that there are two distinct methods for detecting anisotropism in birefracting, transparent media by means of vertically incident light; either intensity differences between the two components parallel and normal to the plane of incidence may be utilized (equation 31) or the rotation of the plane of polarization (equation 34).

Differences of intensity are detected ordinarily by photometric

methods which involve direct comparison of two adjacent illuminated fields (half shade principle). The accuracy of a single setting obtainable with such methods is dependent on the sensitiveness of

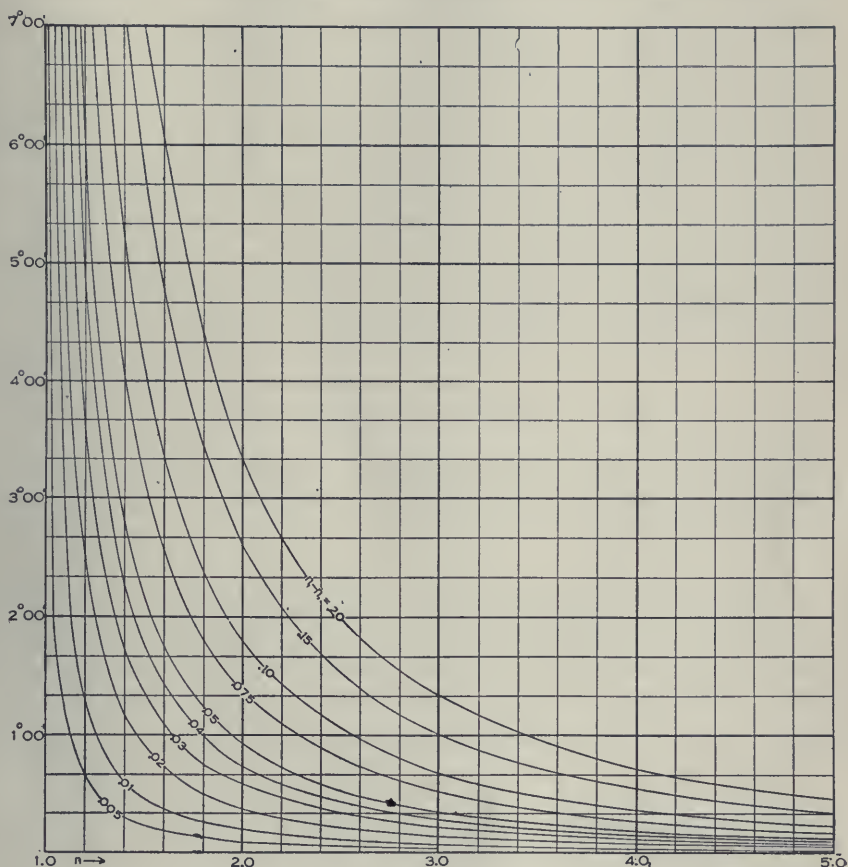


FIG. 3. In this figure is indicated the angular rotation in degrees (ordinates) of the plane of polarization of vertically incident, plane-polarized light of azimuth $\epsilon = 45^\circ$ on reflection from a crystal plate of least refractive index n_1 (abscissæ) and of birefringence $n_2 - n_1$ (curves).

the eye, or on the least perceptible difference in light intensity under the conditions of observation. An extended series of measurements by Koenig and Brodhun³ on the least perceptible increment (dif-

³ *Sitzungsberichte d. Acad. d. Wissensch. Berlin*, July 26, 1888. "Gesammelte Abhandl. und Physiolog. Optik" (A. König), 116-143.

ference limen) for different intensities and different wave-lengths shows that for an intermediate intensity of illumination differences of intensity 1.6 to 2.0 per cent. can just be detected; for low (below 50 m.c.) and high (above 100,000 m.c.) intensities, the least perceptible difference in intensity is greater. Under ordinary conditions of illumination we may assume that 2 per cent. difference in illumination intensity is about the limit perceptible to the eye. This means that in order to be recognized by the eye there must be a difference of about 2 per cent. in intensity of light reflected by the two components. To find the birefringence ($n_2 - n_1$) required to produce this least perceptible difference in intensities of reflected light, namely, 2 per cent., we may without sensible error use only the first two terms of equation (31) and transform it to read

$$n_2 - n_1 = \frac{0.02 (n_1 + 1)(n_2 - 1)}{4}, \quad (35)$$

from which the necessary birefringence ($n_2 - n_1$) for the different refractive indices n_1 can be computed. These are listed in Table 5.

TABLE 5.

In this table is given the least degree of birefringence of a crystal plate which on the reflection of vertically incident light produces a detectable difference in intensity of the two waves polarized at planes normal one to the other.

n_1	$n_2 - n_1$	n_1	$n_2 - n_1$	n_1	$n_2 - n_1$
1.1	.001	1.6	.007	2.5	.026
1.2	.002	1.7	.009	3.0	.040
1.3	.004	1.8	.011	4.0	.075
1.4	.005	1.9	.013	5.0	.120
1.5	.006	2.0	.015	10.0	.440

This table shows that for very low-refracting, non-absorbing substances only a relatively slight increase in refractive index (birefringence weak) is necessary to produce the required perceptible difference in intensity; for minerals of medium refringence and birefringence, such as quartz, the difference limen increases, but the birefringence required is still medium to weak; for highly refracting media the required difference in refractive indices increases rapidly. The essential feature to note, however, is that minerals of medium

refractive index but weak birefringence, such as apatite, do not exhibit the required difference in intensity (2 per cent.) of the reflected components; and that, if the observations were limited to this phenomenon, the anisotropism of the mineral would escape detection. In other words the sensitiveness of methods based on the relative intensity of the two reflected components is far below that for detecting anisotropism in transmitted light. Thus it is not difficult in ordinary thin sections (0.02 mm. thick) to detect a path difference of $2\ \mu\mu$; this corresponds to a birefringence of 0.0001. For a mineral of refractive index 1.6 the least detectible birefringence by reflected intensities is about 0.070. In other words the methods of transmitted light are 50 or more times as sensitive as the methods based on intensity differences of reflected intensities. This follows as a direct result of the lack of sensitiveness of the eye in detecting differences in intensity of adjacent fields.

Methods based on equation (34) depend on the ability of the eye to determine the position of complete darkness (intensity of illumination = 0); the chief factor which limits the degree of precision attainable by this method is the threshold limit of vision or the least quantity of light which the eye can detect. Experience has shown that it is not difficult to detect, under ordinary conditions of illumination with the aid of certain devices, a rotation of the plane of polarization through $5'$. Substituting this value in equation (34) we obtain

$$n_2 - n_1 = 0.0015(n_1 + 1)(n_2 - 1). \quad (36)$$

A comparison of this equation with (35) shows that this method is at least three times as sensitive as the first. The precision attainable by the second method is dependent, moreover, on the sensitiveness of the device employed to detect a rotation of the plane of polarization. Without any special device an error of $\frac{1}{2}^\circ$ is readily possible. In this case the accuracy of this method is considerably less than that of the intensity difference method. In case a device is used which shows a perceptible difference for a rotation of $\frac{1}{4}^\circ$ the precision attainable by the two methods is identical. The accuracy of the second method can, however, be increased by using a very intense source of light. The several methods for determining

positions of extinction have been discussed in detail by the writer⁴ and the conclusion reached that the bi-quartz-wedge-plate is the most sensitive device available for the purpose. With it the sensitiveness can be varied to meet the conditions of illumination. M. Berek⁵ has shown that with the bi-quartz-wedge-plate a precision of 0.5' can be attained under the most favorable conditions of observation.

ABSORBING MEDIA.

In the foregoing section the phenomena produced on reflection from transparent crystal plates are treated in some detail and the factors underlying the methods for the detection of anisotropism and for the measurement of the birefringence and the refractive indices are discussed. The introduction of the absorption index into the equations carries with it a series of complications which render the relations less easy to follow, but which are, as a result, the more interesting.

TABLE 6.

In this table the intensity is given of the light reflected, at vertical incidence, from the surface of an isotropic substance of refractive index n and absorption index κ . Thus a plate of refractive index 2.0 and absorption index 1.32 reflects 50 per cent. of the incident light.

I_{ra}/I_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
n									
0.1	—	—	—	—	—	—	3.51	8.9	16.7
0.2	—	—	—	—	—	3.74	6.06	8.0	12.8
0.3	—	—	.52	1.86	2.00	3.82	5.07	6.92	10.7
0.4	—	.5	1.43	2.17	2.81	3.57	4.67	6.13	9.35
0.5	—	1.0	1.56	2.08	2.78	3.32	4.21	5.57	8.43
0.75	.69	1.1	1.47	1.97	2.29	2.79	3.51	4.70	6.92
1.0	.67	1.0	1.31	1.63	2.00	2.45	3.05	4.00	6.00
2.0	—	.5	.78	1.04	1.32	1.66	2.10	2.78	4.22
3.0	—	—	.36	.67	.94	1.25	1.63	2.21	3.41
4.0	—	—	—	.32	.66	.97	1.33	1.85	2.90
5.0	—	—	—	—	.40	.75	1.11	1.60	2.56
7.5	—	—	—	—	—	.21	.69	1.17	2.03
10.0	—	—	—	—	—	—	.35	.89	1.67
For $\kappa = 0$									
n_1	.52	.38	.29	.23	.17	.13	.09	.06	.03
n_2	1.93	2.62	3.43	4.44	6.04	7.90	11.3	17.9	38.2

⁴ *Am. Jour. Sci.* (4), 26, 377-378, 1908. Carnegie Institution of Washington, Publication 158, p. 139, 1911.

⁵ *Neues Jahrbuch für Mineralogie* etc., Beilage Band 33, pp. 583-661, 1912.

Isotropic, Absorbing Media.—For such substances $n_1 = n_2$, $\kappa_1 = \kappa_2$. The intensity of the reflected light is (equation 22)

$$\frac{Irs}{Is} = \frac{(n - 1)^2 + n^2\kappa^2}{(n + 1)^2 + n^2\kappa^2}. \quad (37)$$

In order to form a clear picture of the relative influence of the

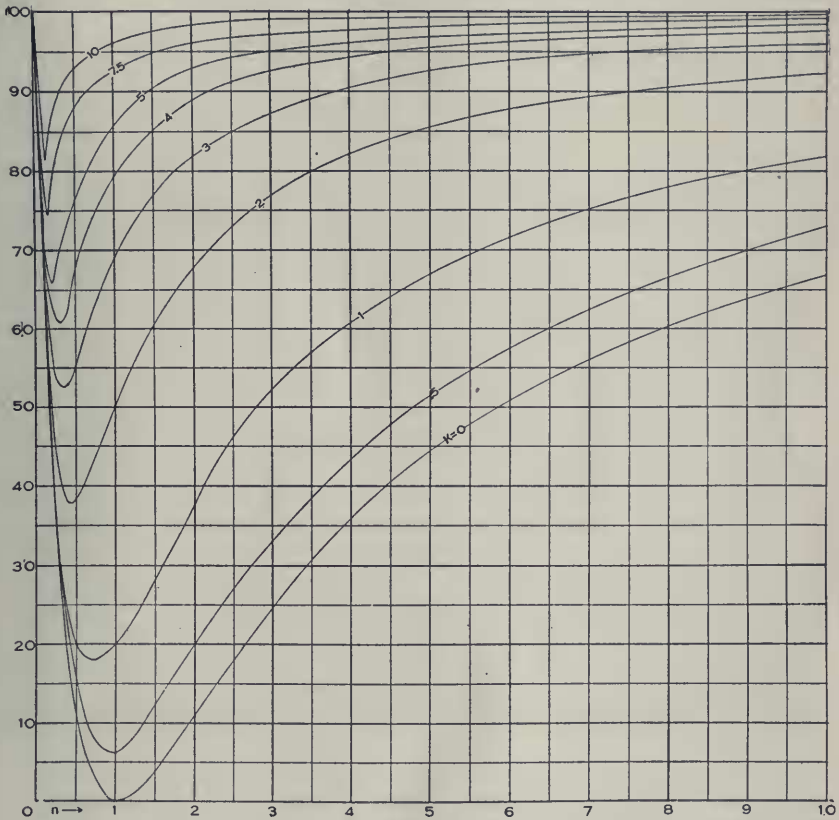


FIG. 4. The percentage (ordinates) of normally incident light reflected from a plate of refractive index n (abscissæ) and absorption index κ (curves) is given in this figure.

refractive index and the absorption index on the reflecting power of an anisotropic, absorbing crystal surface, a series of values computed by means of equation (37) are listed in Table 6 and are rep-

because it shows that high reflecting power (metallic or sub-metallic luster) may arise either from high absorption index as in Na, Ag; or from high refractive index, as in Si and galena; or from both high refractive index and high absorption index, as in Ir, Zn. The curves of these figures show that in general an increase in the absorption index is much more effective in increasing the reflecting power of the substance than the same amount of increase in the refractive index.

The ratio of the amplitude components R_s, R_p is (equation 27)

$$\frac{R_s}{R_p} = -\frac{E_p}{E_p}.$$

For $E_p = E_s$ this reduces to -1 and indicates a phase difference of π between the incident and reflected waves. The phase difference between the two components is (equation 29)

$$\tan (\Delta_2 - \Delta_1) = 0.$$

Therefore plane-polarized, vertically incident light is still plane-polarized after reflection from an isotropic absorbing body.

Birefracting, Biabsorbing Media.—The intensity ratio of the reflected components from a birefracting, biabsorbing crystal plate is stated by equation (28) which can be written for $E_s = E_p$ or $\tan \epsilon = 1$

$$\frac{I_{rs}}{I_{rp}} = 1 - \frac{4[(n_1 - n_2)(1 + n_1 n_2) + n_1 n_2(n_2 \kappa_2^2 - n_1 \kappa_1^2)]}{[(n_1 + 1)^2 + n_1^2 \kappa_1^2][(n_2 - 1)^2 + n_2^2 \kappa_2^2]}. \quad (38)$$

This equation shows that for weakly birefracting substances (n_1 approximately equal to n_2) the intensity ratio depends on the difference of the squares of the absorption indices. For substances with weak biabsorption the intensity ratio depends primarily on the difference of the refractive indices. This indicates that an increase in the biabsorption is more effective in increasing the degree of anisotropism than the same amount of increase in birefringence.

The phase difference between one of the components and that of the incident beam is given by equation (23). Series of values computed by means of this equation are listed in Table 7 and are presented in graphical form in Fig. 6 in which the refractive indices

are the abscissæ, the ordinates the angles ($\tau - \Delta_1$), and the curves the absorption indices. The curves show clearly that for substances with high refractive indices and high or low absorption indices the

TABLE 7.

In this table are listed the phase differences between the vertically incident beam and one of the components reflected from a plate of refractive index n and absorption index κ . Thus for a plate of refractive index 1.5 and absorption index 0.2, the phase difference is $24^\circ 07'$.

κ	0.2	0.4	0.6	0.8	1.0	1.5
n						
0.2	$-4^\circ 47'$	$-9^\circ 32'$	$-14^\circ 14'$	$-18^\circ 54'$	$-23^\circ 52'$	$-34^\circ 36'$
0.4	$-10 52$	$-21 27$	$-31 31$	$-41 00$	$-49 37$	$-68 12$
0.6	$-21 00$	$-39 30$	$-54 40$	$-66 52$	$-76 52$	$85 15$
0.8	$-45 30$	$-68 03$	$-82 18$	$87 47$	$80 05$	$65 46$
1.0	$84 18$	$78 42$	$73 18$	$68 12$	$63 27$	$53 08$
1.5	$24 07$	$36 42$	$41 10$	$41 44$	$40 36$	$35 30$
2.0	$14 13$	$23 45$	$28 24$	$29 55$	$29 45$	$26 34$
3.0	$8 10$	$14 16$	$17 45$	$19 13$	$19 36$	$18 26$
4.0	$5 51$	$10 19$	$13 01$	$14 13$	$14 28$	$13 14$
5.0	$4 35$	$8 08$	$10 18$	$11 19$	$11 32$	$10 36$
7.5	$2 59$	$5 20$	$6 49$	$7 29$	$7 40$	$7 03$
10.0	$2 01$	$3 59$	$5 04$	$5 36$	$5 45$	$5 18$

κ	2.0	3.0	4.0	5.0	7.5	10.0
n						
0.2	$-45^\circ 00'$	$-63^\circ 27'$	$-78^\circ 42'$	$88^\circ 51'$	$66^\circ 43'$	$52^\circ 45'$
0.4	$-82 53$	$75 58$	$61 44$	$51 41$	$36 20$	$27 49$
0.6	$71 34$	$54 11$	$43 11$	$35 41$	$24 40$	$18 45$
0.8	$55 30$	$41 37$	$32 55$	$27 05$	$18 36$	$14 06$
1.0	$45 00$	$33 41$	$26 34$	$21 49$	$14 56$	$11 19$
1.5	$30 21$	$22 43$	$17 51$	$14 38$	$9 59$	$7 33$
2.0	$22 50$	$17 06$	$13 26$	$11 00$	$7 30$	$5 40$
3.0	$15 16$	$11 30$	$8 36$	$7 20$	$5 00$	$3 49$
4.0	$11 27$	$8 35$	$6 44$	$5 31$	$3 45$	$2 50$
5.0	$9 10$	$6 52$	$5 24$	$4 25$	$3 00$	$2 16$
7.5	$6 07$	$4 35$	$3 36$	$2 56$	$2 00$	$1 31$
10.0	$4 35$	$3 27$	$2 42$	$1 55$	$1 30$	$1 08$

phase difference is only a few degrees and therefore, so far as useful in measurements, is practically negligible. In other words the reflected light, although elliptically polarized, approaches plane polarized light in character and the essential phenomena to be observed are intensity differences and rotation of the plane of vibration. This statement is valid for all substances of refractive index above 3.0 and for substances of still lower refractive index (1.5) provided the absorption index is either high or very low (less than 0.3). The

region for which slight changes in refractive index or absorption index produce the greatest difference in phase are for low refractive indices $n < 1.5$ especially for $n < 1$ and for high absorption indices. Substances, however, with such indices are rare; the conclusion is therefore generally valid that the phase difference between the two components of a birefracting and biabsorbing substance for ver-

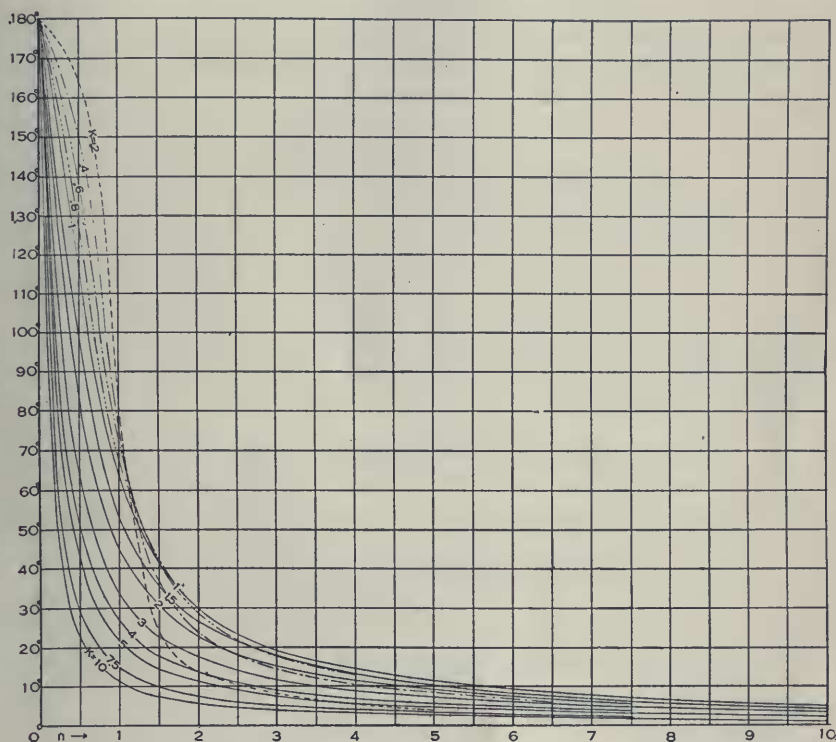


FIG. 6. In this figure is given the phase difference (ordinates) between one of the components of normally incident polarized light (azimuth $\epsilon = 45^\circ$) and the same component after reflection from a birefracting, biabsorbing crystal plate of refractive index n (abscissæ) and absorption index κ (curves).

tically incident light is not great and that the anisotropism finds expression chiefly in the intensity differences of the two components. This conclusion is important, since it enables the observer to apply the methods outlined above for non-absorbing media to the detection of anisotropism in opaque substances.

The amplitudes Rs , Rp (equation 20, 24) are complex and of the form $f + ig$. In order to compound these two vibrations (vectors) and to ascertain the directions of the resultant elliptical vibration let (equation 21)

$$\begin{aligned}Rs &= r_1 e^{i(\tau - \Delta_1)}, \\Rp &= r_2 e^{i(\tau - \Delta_2)}.\end{aligned}\tag{39}$$

If ψ be the angle between the Rp axis and the major axis of the resultant ellipse of vibration, then we find by taking the real parts of (39), namely

$$\begin{aligned}Rs &= r_1 \cos(\tau - \Delta_1), \\ \text{and } Rp &= r_2 \cos(\tau - \Delta_2),\end{aligned}\tag{40}$$

and compounding them into the form

$$\begin{aligned}u_0 &= r_0 \cos(\tau - \Delta_0), \\ v_0 &= r_0 \sin(\tau - \Delta_0),\end{aligned}\tag{41}$$

which obtains for the components after the principal axes of the elliptical vibration, that

$$\tan 2\psi = \frac{2r_1 r_2}{r_1^2 - r_2^2} \cos(\Delta_1 - \Delta_2).\tag{42}$$

To effect this transformation the normal transformation equations

$$\begin{aligned}u_0 &= u \cos \psi + v \sin \psi, \\ v_0 &= -u \sin \psi + v \cos \psi,\end{aligned}\tag{43}$$

are used and the coefficients of $\cos \tau$ and $\sin \tau$ of equations (41) and (43) are equated.

If we put

$$\frac{r_2}{r_1} = \tan \vartheta,\tag{44}$$

equation (42) may be written

$$\tan 2\psi = \tan 2\theta \cos(\Delta_1 - \Delta_2).\tag{45}$$

The quantities on the right hand of this equation can be computed from equations (27), (29), (40) and (41), (44); but the expression thus obtained is too complicated to be readily interpreted. The fact that the phase difference for most opaque substances is, as

shown above, small and therefore the cosine of the angle ($\Delta_2 - \Delta_1$) not greatly different from unity indicates that ψ is only slightly smaller than θ whose tangent is (equations (22) and (44))

$$\tan^2 \vartheta = \frac{r_2^2}{r_1^2} = \frac{[(n_2 - 1)^2 + n_2^2 \kappa_2^2][(n_1 + 1)^2 + n_1^2 \kappa_1^2]}{[(n_2 + 1)^2 + n_2^2 \kappa_2^2][(n_1 - 1)^2 + n_1^2 \kappa_1^2]}.$$

The right-hand side of this expression is identical with that obtained for the intensity ratio. Therefore, in general, the square of the tangent of the angle of rotation θ is approximately equal to the intensity ratio. As this increases so also does the angular rotation of the plane of polarization.

The above equations and the methods based on these equations do not suffice for the determination of the optical constants n_1 , n_2 , κ_1 , κ_2 of an opaque crystal examined only under vertical incidence. They do indicate, however, that for the detection of anisotropism the phase difference between the two components R_s and R_p is ordinarily not sufficiently large to be readily measured and that therefore the* phase difference is of little value as a diagnostic feature. The other variable factor is the amplitude ratio. This gives rise to a difference in intensity of the components of the reflected wave. For certain cases in which the reflected waves are plane-polarized one of the components is more intense than the second and the excess in intensity in the one direction produces a certain amount of polarized light after reflection of the non-polarized, incident waves; this can be detected by methods similar to those which have long been in use for the detection and measurement of polarized light in the sky. Since for most substances the phase difference between the reflected components is not great, methods suitable for the detection of small angular rotations of the plane of polarization of incident plane-polarized light may also be used to detect anisotropism in opaque bodies. A brief description of the several methods which may serve for this purpose will now be given.

METHODS BASED ON INTENSITY CONTRAST.

These methods are widely used in the measurement of sky polarization and have proved to be of great usefulness in that connection. One of these methods has been applied by J. Koenigsberger to the detection of anisotropism in plates of opaque crystals.

*Koenigsberger's Method.*⁶—In a series of articles Koenigsberger suggested the use of a Savart plate⁷ in conjunction with an analyzing nicol and a telescope for the detection of polarization in light

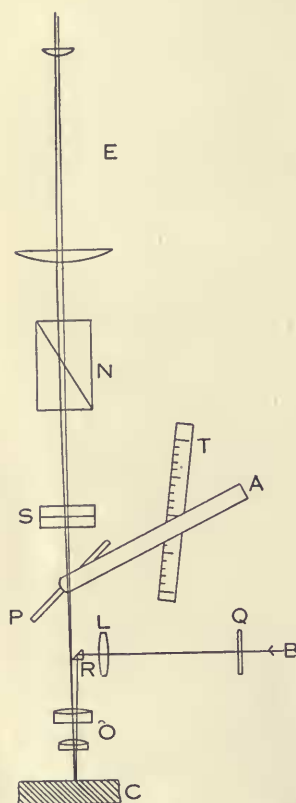


FIG. 7. Koenigsberger's apparatus for the detection and measurement of anisotropism in opaque substances.

reflected from a crystal surface, the incident light to be non-polarized and to impinge vertically on the plate. The arrangement proposed by Koenigsberger is illustrated in Fig. 7. The non-polarized light (monochromatic or white) enters the system along *B*, passes through a contrast plate *Q* consisting of a bi-plate of smoky quartz cut parallel to the axis and mounted so that the axis is at right angles in the adjacent halves; a lens *L*, for imaging the contrast plate in the image plane of the telescope *E*. The light passes through *L* to a small reflecting prism *R*, thence through the objective *O* to the crystal plate *C* where it is reflected back through *O* past *R*, through a plane-parallel, rotatable glass plate *P* of known refractive index ($n=1.515$), through the Savart plate *S* and the nicol *N* into the telescope *E*.

The Savart plate consists of two plates of calcite or quartz cut at an angle with the optic axis sufficiently large (45°) that if examined in convergent polarized light the isochromatic curves of the interference figure cross the field as practically straight lines. The two plates are of equal thickness and are superimposed so that the horizontal projection of the axis in the one is at right

⁶ *Centralblatt für Mineralogie, Geologie u. Paläontologie*, 1901, pp. 195-197; 1908, pp. 565-569, 597-605; 1909, pp. 245-250; 1910, pp. 712-713.

⁷ Devised and first described by D. Brewster, *Edinburgh Transactions*, 9, 148, 1819; described later by Savart, *Pogg. Ann. d. Phys.*, 49, 292, 1840.

angles to that of the axis in the second. The relations are shown in Fig. 8. If the combination be observed between crossed nicols with the axes at 45° with the principal nicol planes, a series of parallel, vertical dark interference bands appears, if monochromatic light be used, in the field; if a white light source be used the bands, except one central band, are colored. These bands are similar in appearance to the bands in a quartz wedge or a Babinet compensator; but their mode of formation is different as they depend on very slightly convergent polarized light for their development while in the quartz wedge the change in thickness of the wedge introduces the required

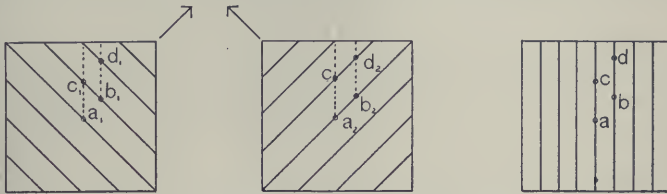


FIG. 8. Diagrams illustrating the principle on which the construction of the Savart plate is based.

path difference. From the mode of formation it is evident that if non-polarized light be used no such bands will appear, but that with a small percentage of polarized light there will be superimposed on the white field a series of colored bands the intensity of which increases with the amount of polarized light present. Under the best conditions of setting at the center of the light bands the maximum intensity is obtained, namely, $\frac{1}{2}$ of total non-polarized light and $\frac{1}{2}$ of total polarized light; whereas at the center of the dark central band only the non-polarized portion is transmitted. At intermediate points the non-polarized light and a part of the polarized component is transmitted. The field, therefore, alternates in intensity; the least amount of polarized light which can be detected depends obviously on the least perceptible increment in intensity which the eye is able to detect. As noted above Koenig and Brodhun's data place this difference limen at 1.6 to 2 percent. for favorable intensity of illumination. Pickering and others⁸ estimate that in sky polarization about 1 per cent. of polarized light can be detected. Koenigsberger

⁸ Report of U. S. Naval Observatory of Total Eclipse of July 29, 1878.

asserts that 0.3 per cent. difference can be detected, but a series of experiments by the writer using a rotating, optically plane-parallel plate and different sources and intensities of illumination indicate that the figures of Koenig and Brodhun are more nearly correct and that Koenigsberger's statement of the precision attainable is about 5 times too great. The difference between different settings, especially for large intensity differences, is of course considerably less than the least perceptible increment but this difference may not be considered to be the least perceptible increment itself.

In order to render more readily visible the Savart bands and the point of exact compensation Koenigsberger employs a contrast bi-plate of smoky quartz which introduces a difference in intensity between the components (result of pleochroitic absorption) and thus produces a shift of the lines in the halves of the field; these lines are then shifted by the changes in intensity resulting on reflection from an anisotropic crystal plate.

In the practical application of this method it is essential that the plate to be examined be well polished and normal to the axis of the microscope; that the reflecting prism be not too large; that the rotating glass plate *P* be mounted with its axis at 45° with the upper nicol plane; that the Savart plate be accurately constructed and be normal to the microscope axis; that the telescope be accurately focussed on infinity (in order that convergent light of only very small angular aperture pass through the Savart plate). To measure the degree of anisotropism the glass plate *P* is rotated until the effect of the crystal plate is exactly compensated and the Savart bands disappear or are separated by exactly half a band if the contrast band be used.

The effect of the tilted plate in compensating intensities normal and parallel with the plane of symmetry can be found from equations 15 simplified for the case of an isotropic body. Thus if *D* be the amplitude of the transmitted plane-polarized wave and δ its azimuth, we find

$$D \cos \delta = E \cos \epsilon \frac{\sin 2r \sin 2i}{\sin^2(i + r)} = E \cos \epsilon \cdot C_1,$$

$$D \sin \delta = E \sin \epsilon \frac{\sin 2r \sin 2i}{(\sin i \cos i + \sin r \cos r)^2} = E \sin \epsilon \cdot C_2.$$

The intensity contributed by the component of the wave D (azimuth δ) in the plane of incidence is accordingly

$$D^2 \cos^2 \delta \cdot d\epsilon = E^2 \cos^2 \epsilon \cdot C_1^2 d\epsilon.$$

The total intensity for the components of all waves D (non-polarized light) in the plane of incidence is

$$E^2 C_1^2 \int_0^{\pi/2} \cos^2 \epsilon \cdot d\epsilon.$$

Similarly the total intensity for the components of all waves D (non-polarized light) normal to the plane of incidence is

$$E^2 C_2^2 \int_0^{\pi/2} \sin^2 \epsilon \cdot d\epsilon.$$

The ratio of the two intensities is

$$\frac{Ip}{Is} = \frac{C_1^2}{C_2^2} = \frac{(\sin i \cdot \cos i + \sin r \cdot \cos r)^4}{\sin^4 (i + r)} = \cos^4 (i - r). \quad (46)$$

The same expression can be derived more directly from equation (16) if we consider the waves to pass through the system in reverse direction which is permissible. A series of values computed by means of this equation is listed in Table 8 and shown by the curve of Fig. 9.

TABLE 8.

Let a plane-polarized light beam be incident at the angle i on a plane parallel glass plate of refractive index 1.515; let the plane of polarization include angle of 45° with the plane of incidence. The ratios of the intensities of the two components, parallel and normal to the plane of incidence, of the beams emerging from the plate under these conditions are given in this table.

i	$I_p/I_s = \cos^4 (i - r).$	i	$I_p/I_s = \cos^4 (i - r).$
0°	1.0000	25	67.47
5	.9848	30	56.25
10	.9406	35	45.02
15	.8705	40	34.51
20	.7798	45	25.00

In this figure the curve drawn by Koenigsberger and furnished with his apparatus is reproduced as the dotted curve. (Refractive index

of glass plate = 1.515). The two curves do not coincide; no satisfactory explanation for the discrepancy has yet been found.

Photometrically the gradation in intensity between the different Savart bands is less advantageous than the juxtaposition of two or a series of evenly illuminated fields, the one at minimum intensity the next at maximum intensity. This principle is used in measurements of sky polarization and can be readily applied to the present problem.

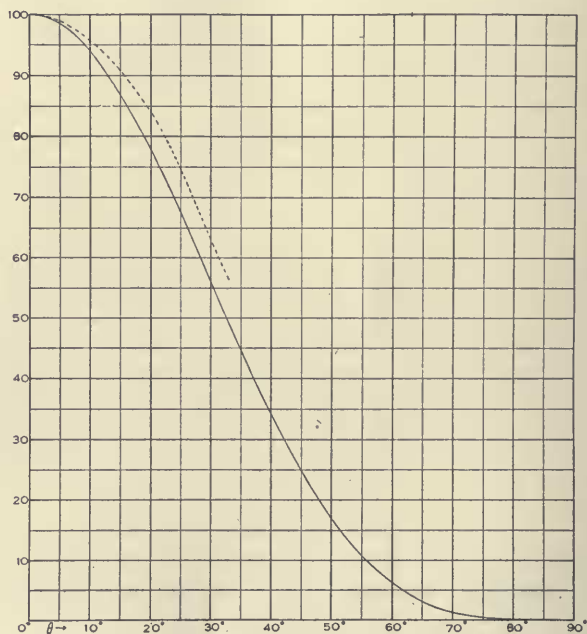


FIG. 9. Curve illustrating the percentage intensity ratio (ordinates) between the two components, normal and parallel to the plane of incidence, of light transmitted through a tilted glass plate for different angles of rotation (abscissæ).

New Method.—For this method use is made of a cleavage plate of calcite of such thickness and an aperture of such width that the two fields from the two rays just touch (Fig. 10); better fields are obtained by use of a small Koenig-Martens portable photometer⁹ (Fig. 11).

⁹ *Verhandlungen d. deutsch. Physikal. Gesellschaft*, 1, 204-208, 1899; *Physikalische Zeitschrift*, 1, 299-303, 1900.

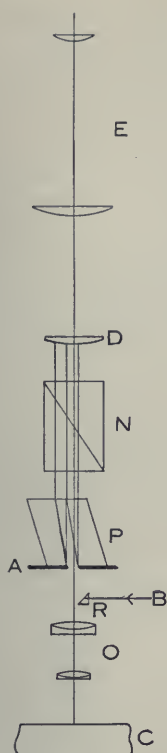


FIG. 10. New method for the detection and measurement of the degree of anisotropy of a crystal plate. In this method the cleavage plate of calcite serves to produce two adjacent fields whose planes of polarization are normal to each other. The intensity of the illumination of the two fields is made equal by rotation of the nicol *N*.

polarized light be present the intensity of illumination of the two fields is different; but it can be made equal by rotating the nicol *N*

In Fig. 10 the non-polarized incident light is reflected by the prism *R* to the plate *C* whence it passes on reflection through the calcite plate aperture *A* to the nicol *N*, the weak lens *D* and the microscope eyepiece *E*. In the writer's microscope the upper nicol can be rotated and the angle of rotation read off to ϕ' on the stage of the microscope. The function of the low-power lens *D* is to image the aperture *A* in the focal plane of the weak eye lens *E*.

In Fig. 11 the calcite plate is replaced by the Koenig-Martens arrangement of aperture *A*, field lens *D* (to render rays from aperture *A* to Wollaston prism *W* parallel), and the twin prism *F*. This gives a better photometric field than the calcite cleavage plate. In both methods artificial light, non-polarized, and either monochromatic or white is used and rendered diffuse either by the interposition of a ground glass or opal glass plate. The fields should be equally illuminated and no junction line between them should be visible. In case

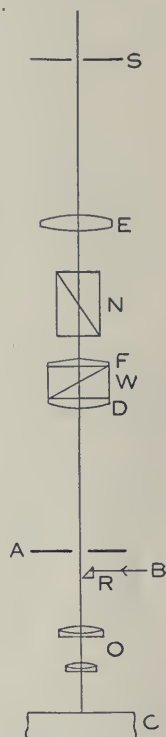


FIG. 11. In this method the Koenig-Martens arrangement of wollaston prism and twin glass prism is substituted for the calcite cleavage plate of Fig. 10. The intensity of the illumination of the two adjacent fields is rendered equal by rotation of the nicol *N*.

through an angle α ; in this case the observed intensity from the one field of intensity I , is $I \cos^2 (45^\circ - \alpha)$; that of the second, I_2 , is $I \cos^2 (45^\circ + \alpha)$. Since the observed intensities are equal we have

$$\frac{I_2}{I_1} = \frac{\cos^2 (45^\circ + \alpha)}{\cos^2 (45^\circ - \alpha)} = \tan^2 (45^\circ - \alpha). \quad (47)$$

A series of values computed by means of this equation is listed in Table 9. For practical purposes a large-scale curve plotted on millimeter cross section paper is useful for ascertaining intermediate values.

TABLE 9.

In this table are listed the relative intensities of the sources of illumination of the two fields of a calcite rhomb or a Koenig-Martens photometer for different settings α of the nicol N at which equal field intensities are obtained.

α	$\tan^2 (45 - \alpha)$	α	$\tan^2 (45 - \alpha)$	α	$\tan^2 (45 - \alpha)$
0°	1.0000	7°	.6104	18°	.2596
1	.9326	8	.5678	20	.2174
2	.8696	9	.5279	25	.1325
3	.8107	10	.4903	30	.0718
4	.7557	12	.4217	35	.0311
5	.7041	14	.3610	40	.0077
6	.6558	16	.3073	45	.0000

Of the two devices the Koenig-Martens photometer is the better because it gives a better photometric field and is an attachment complete in itself which may be used on any microscope in place of the eyepiece. The only additional accessory required is the reflecting prism or mirror with which every metallographic microscope is equipped.

These methods are superior to the Koenigsberger method, not only because of increased sensitiveness, but also because the particular plate under investigation can be viewed directly (at high or low magnifications) and the test made directly on the plate in full view. In the Koenigsberger method the Savart bands are not readily distinguished unless the plate is brought out of sharp focus and even then the irregularities of the image tend to render indistinct and uncertain the faint Savart bands.

METHODS BASED ON THE DETECTION OF A ROTATION OF THE PLANE OF POLARIZATION
(POSITION OF EXTINCTION).

These methods have been long in use by petrologists and the principles underlying the construction of the different devices need not be repeated here.¹⁰

*Koenigsberger's Method.*¹¹—In this method Koenigsberger adopted the arrangement shown in Fig. 12. The incident light passes first through the polarization prism (vibration plane horizontal or vertical) *P*, and the total reflecting prism *R* to the crystal plate whence it travels through the Biot quartz plate *A* (cut normal to the axis and of such thickness, 3.75 mm., that it gives the sensitive tint between crossed nicols), the eyepiece *E*, the cap nicol *N* to the eye of the observer. The rotation of the plane of polarization on reflection is detected by the change in interference hue on rotating the crystal plate. Koenigsberger emphasizes the fact that this method is not so sensitive as his first method and is only qualitative in nature.

Hanemann's Method.—In 1913 Hanemann¹² improved Koenigsberger's method by substituting a Soleil-Biot plate (two adjacent Biot quartz plates, 3.75 mm. thick, cut normal to the axis, the one of right-handed, the second of left-handed quartz) for the single Biot plate. This arrangement introduces the feature of color con-



FIG. 12. Diagram showing general arrangement for the detection of the rotation of the plane of polarization of normally incident plane-polarized light waves on reflection from a birefracting and bi-absorbing plate.

¹⁰ These are treated at length in "The Methods of Petrographic Microscopic Research," Carnegie Institution of Washington Publication 158, pp. 115-148, 1911.

¹¹ *Centralblatt für Mineralogie*, etc., 1909, 245-250; 1910, 712-713; *Metallurgie* 4, 605-608, 1909.

¹² K. Endell and H. Hanemann, *Stahl und Eisen*, 1913, p. 40; *Zeitschrift für anorganische Chemie*, 83, 267-274; 88, 265-268, 1914.

trast in adjacent fields and is accordingly more sensitive than the Koenigsberger method.

The Bertrand eyepiece which is in common use by petrologists would serve the purpose as well as, and possibly better than, the Soleil-Biot plate employed by Hanemann.

New Method.—The methods of Koenigsberger and of Hanemann are based on changes in color (sensitive tint) on rotation of an anisotropic plate. In the case of strongly colored substances, especially yellow minerals, the natural color dominates the field so effectively that the sensitive tint is no longer present as such and the slight changes in hue on rotation of the stage are only with difficulty detected and may be overlooked. The sensitiveness of these methods varies therefore with the color of the substance under examination. It is also difficult to obtain a uniformly colored field under average conditions of illumination, especially near the junction lines in the field.

In measurements of this nature in which the effective conditions vary within wide limits, whereas the sensitiveness of the eye (threshold vision) is more or less fixed it is essential for the best results that the sensitiveness of the device employed for the measurement be variable so that the most sensitive conditions of observation can be obtained. It is on this principle of variable sensibility that the writer's bi-quartz-wedge-plate¹³ was constructed.

On substituting the bi-quartz-wedge-plate for the Biot plate of Koenigsberger or the Biot-Soleil plate of Hanemann the most sensitive arrangement for the detection of anisotropism in opaque substances is obtained. The method is exceedingly simple and requires no apparatus in addition to that furnished with a research model petrographic microscope. The observations are made either in white or in monochromatic light. The degree of anisotropism is indicated by the amount of angular rotation of the cap nicol required to produce equal intensity of illumination in the adjacent halves of the bi-quartz-wedge-plate.

Other devices, such as the half-shade Lippich prism, may be substituted for any one of the rotating quartz plates and wedges but

¹³ *Am. Jour. Sci.* (4), 26, 377-378, 1908; Carnegie Institution of Washington, Publication 158, 140-142, 1911.

since the apparatus is more complicated and the attainable accuracy is not increased, such devices are not recommended for practical work.

Effect of Reflecting Prism or Plate on the Character of Light.—

An extended series of experiments with different kinds of reflectors, total reflecting prisms, silver surfaces, silvered glass mirrors with central aperture or a small disk mirror in center of field, or an Abbe reflecting cube, was carried out to ascertain if possible which type is the best for such work, especially for use with the second group of methods. The general equations state that in case the azimuth of the plane of polarization of the incident wave is zero or 90° , the reflected beam is still plane polarized; practical experience with such reflectors states, however, that the reflected beam always shows traces of elliptical polarization even when the plane of polarization is horizontal or vertical. This is no doubt due in the case of reflecting glass surfaces to internal reflections; with a blank metal reflecting surface, such as the fresh silver side of a silvered mirror, it may result from strains in the outer film. Be the cause what it may, in no experiment was the elliptic polarization entirely removed, and the accuracy of the settings was correspondingly diminished. Experience with the different types of reflectors did not demonstrate marked superiority of any one particular type. It is, of course, essential that the plane of polarization of the incident beam be strictly horizontal or vertical in order to reduce to a minimum the amount of elliptic polarization present.

MEASUREMENT OF PHASE DIFFERENCES IN REFLECTED WAVES.

The curves of Fig. 6 prove that in general the phase difference between the components of the reflected waves normal and parallel to the plane of incidence is of a low order of magnitude, so low in fact that it may in general be neglected. The actual measurement of the phase difference is best accomplished by the standard method with a Babinet compensator. In view of the slight differences to be observed, however, this method is not to be recommended for practical diagnosis.

The Effect of Thin Surface Films on the Character of the Reflected Light.—Drude¹⁴ was the first to emphasize the importance of thin surface films in affecting the polarization relations of reflected and of transmitted light waves. In the case of opaque substances (ores and metals) the surface film effects may be serious and cause an isotropic mineral to show evident phenomena of anisotropism. Thus a sheet of copper or other isotropic metal shows distinct anisotropic effects. The rolled surface of the metal is striated and part of the polarization effects may be of the nature of grating polarization effects or may result from reflection at oblique surfaces. Many ground surfaces of isotropic pyrite are decidedly anisotropic. In certain cases this is probably due, as Koenigsberger suggested¹⁵ to grinding striæ; but in other pyrite sections the anisotropism persists in spite of care taken to eliminate grinding striæ. This may be the result of surface film effects or of strain birefringence in the pyrite. Its presence in random sections, even carefully ground, serves as a danger-signal, warning the observer against the drawing of too positive conclusions from a single section. As a general rule pyrite sections are isotropic and the conclusion is warranted that it is isometric, as crystallographic measurements prove it to be. The statement of Koenigsberger that surface films affect only the phase difference and not the amplitudes of the reflected beams is borne out neither by theory nor by practical experience. In general, however, it may be stated that surface film effects are not so pronounced as to affect seriously the validity of the determinations.

The Use of Refractive Liquids.—In case the medium surrounding the crystal be not air but a refractive liquid the refractive indices appearing in the above equations have to be reduced in the ratio of the reciprocal of the refractive index of the liquid; thus if n_0 be the refractive index of the liquid, n_1 , n_2 of the crystal plate must be replaced by n_1/n_0 , n_2/n_0 . With each different refractive liquid employed both the intensity ratio of the reflected components and the angular rotation of the plane of polarization are changed. Theo-

¹⁴ *Ann. d. Physik*, 43, p. 146, 1891; *Lehrbuch d. Optik*, 1906, pp. 272-280; Winkelmann's *Handbuch d. Physik*, Vol. 6, pp. 1278-1316, 1906.

¹⁵ *Centralblatt für Miner., Geol. u. Paläontol.*, 1908, p. 597.

retically it is thus possible by measurements in different immersion liquids to determine the refractive indices n_1 , n_2 , and also the absorption indices κ_1 , κ_2 even with normally incident light; but practically this method is of little value because the results are encumbered with a large probable error due chiefly to the relative insensitiveness of the eye to slight changes in the observed phenomena.

In a transparent, birefracting substance the maximum effect is obtained by the use of a liquid of refractive index equal either to n_1 or to n_2 . The intensity of one of the reflected components is then zero; the intensity of the second component for $n_0 = n_1$ is

$$\frac{I_{rp}}{I_r} = \left(\frac{n_2 - n_1}{n_2 + n_1} \right)^2,$$

which is a very small quantity for weakly birefracting substances; and even for strong birefringence such as that of calcite it amounts only to one half of one per cent. With intense illumination, however, relatively weak birefringence can be detected by the use of proper refractive liquids. Since clear refracting liquids of index above 2.2 are not available, it is not possible to obtain the most sensitive conditions of observation with minerals of very high refractive index, such as hematite and selenium. But with most minerals conditions of observation can be improved somewhat by the use of immersion liquids of proper refractive index. In practical work, however, it is doubtful if the improvement thereby gained is worth the bother.

SUMMARY.

In the foregoing pages the attempt has been made to present in connected form the electromagnetic theory of the reflection of light by absorbing media and especially that part of the theory which treats of the reflection phenomena resulting from vertically incident light waves under the conditions usually encountered in the use of the reflecting or metallographic microscope; this outline of the theory is given in order to indicate and also to emphasize the fundamental principles on which all methods involving the application of polarized light to the study of opaque substances are necessarily based; thus the possibilities and the limitations as well of

such methods are ascertained. It is shown in the general case that vertically incident, plane-polarized light waves become on reflection elliptically polarized (as a result of a difference in phase between the components parallel and normal to the plane of incidence); that the amplitude of the component of the light vector in the plane of incidence is different from that normal to the plane of incidence.

In special cases where crystallographic symmetry relations prescribe the positions of the principal axes of refringence and of absorption, as in isotropic, uniaxial, and the principal planes of orthorhombic crystals, the above relations are simplified to the extent that the refracted waves are plane-polarized and not elliptically polarized as in the general case; as a result, normally incident, plane-polarized light waves whose vibration directions are either parallel or normal to the plane of symmetry are reflected as plane-polarized waves; but the intensity of the two reflected waves is different because of the difference in the refractive and absorption indices parallel and normal to the plane of symmetry. Therefore normally incident, non-polarized light contains after reflection a certain amount of plane-polarized light and this amount increases with the strength of the birefringence and of the biabsorption in the crystal plate.

The presence of plane-polarized light in essentially non-polarized light can be detected by any one of the well-known physical methods, such as are commonly used in determinations of sky polarization. Of these methods Koenigsberger adopted the Savart plate with rotating glass compensator; a second method is suggested above which employs either a single calcite cleavage plate with proper aperture (after the manner of the Haidinger lens or the Pickering photometer) or a small portable Koenig-Martens photometer. This method is superior to the first method in two respects: it is simpler in adjustment and in manipulation; it is based on a photometrically better principle, namely contrast of two or a series of illuminated fields, the first at minimum intensity, the adjacent field at maximum intensity with a sharp line of demarcation which disappears when the intensity of illumination in the adjacent fields is the same. The attainable accuracy depends on the least perceptible difference in intensity of illumination which the eye can detect; this is about 1.5 to 2 per cent. under ordinary conditions of illumination. An opaque

crystal, therefore, of such weak birefringence or weak biabsorption such that the difference in intensity between the reflected components is less than 1.5 per cent. appears isotropic. In transmitted light differences in birefringence of only 0.02 per cent. of the refractive index are readily detectible; therefore the methods based on differences in intensity of the reflected components of vertically incident light are 50 or more times less sensitive in the detection of anisotropism than the methods based on the phenomena of plane-polarized, transmitted light waves.

In case the vertically incident light be plane-polarized, the difference in amplitude of the reflected components normal or parallel to the plane of symmetry causes a rotation of the plane of polarization and this can be detected and measured by any one of a number of devices in common use by petrologists. Of these Koenigsberger adopted a Biot sensitive tint plate and used it in a qualitative way to detect anisotropism. Hanemann improved Koenigsberger's method by adopting the Biot-Soleil sensitive tint bi-plate and obtained thereby a color contrast in two adjacent fields. The ordinary Bertrand eyepiece which is in common use by petrologists is suggested above as a still further and equally simple method for detecting anisotropism.

A still more sensitive and better method is, however, to use the writer's bi-quartz-wedge-plate in which the sensibility is variable and can be adjusted to meet the conditions of illumination and thus to produce the most favorable conditions for extreme sensitiveness and consequent accuracy of results of measurement. This method is simple and is the most sensitive at present available for detecting and measuring anisotropism in opaque substances. The degree of anisotropism is indicated by the amount of angular rotation on reflection of the plane of polarization of the vertically incident light waves. The accuracy of these methods depends on the threshold limit of vision (least intensity of light which the eye can detect); this is different from the least perceptible difference in intensity of illumination of two adjacent fields on which the sensitiveness of the first group of methods depends.

The sensitiveness of the second group of methods can be increased by increasing the intensity of the light source and also by

introducing some contrast device such as the bi-quartz-wedge-plate, or the Lippich biprism. In general it may be stated that for practical purposes the methods of the second group are simpler in manipulation than those of the first and more sensitive and therefore better. A disturbing factor is the introduction of a small amount of elliptically polarized light by the reflecting prism or plate. The phase differences between the two components of the reflected waves are in general small and do not seriously affect measurements by the methods outlined above which are based on the amplitude differences between the two components. The phase differences may be measured by one of several devices of which the Babinet compensator is the best and most widely used; but for the detection and measurement of anisotropism, methods, which are based on phase differences between the components after reflection of vertically incident plane-polarized light, are not in most cases of practical value.

The use of immersion liquids and a monochromatic light source is suggested for obtaining the greatest differences in intensity between the reflected components and thereby increasing the sensitiveness of the methods for detecting anisotropism.

The above equations show that it is in general not possible to measure both the refractive indices and the absorption indices of an opaque body by means of the phenomena resulting on the reflection of vertically incident light waves; but it is possible in special cases to determine the degree of anisotropism and also the positions of the polarization directions in a given crystal plate. These data taken in conjunction with crystallographic data enable the observer, just as do birefringence and extinction angles in transparent crystals, to draw conclusions regarding the crystal system of the substance under investigation. But in opaque substances the precision attainable is relatively slight and the phenomena which can be observed are relatively few and restricted in scope. As a result one cannot expect from the application of polarized light to such substances the harvest of optical data which have been gathered from transparent crystals. For opaque bodies the possibilities are few, the limitations are great, and recourse must be had in practical diagnosis to other methods of determination such as behavior of the substance toward abrasives

and solvents, color, density, hardness, etc., in addition to the determination of the degree of anisotropism.

In this paper the properties of a number of opaque substances such as ores and metals, determined by the several methods described above, are not included; such lists would add materially to its length and are not germane to its purpose.

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